

# Contents

<b>0</b>	<b>Review from Classical Differential and Projective Geometry</b>	<b>1</b>
0.1	Developable Rulings . . . . .	1
0.2	Vanishing Gauß Curvature . . . . .	3
0.3	Hessian Matrices . . . . .	5
0.4	Classification of Developable Surfaces in $\mathbb{R}^3$ . . . . .	7
0.5	Developable Surfaces in $\mathbb{P}_3(\mathbb{C})$ . . . . .	9
<b>1</b>	<b>Grassmannians</b>	<b>12</b>
1.1	Preliminaries . . . . .	12
1.1.1	Algebraic Varieties . . . . .	12
1.1.2	Rational Maps . . . . .	16
1.1.3	Holomorphic Linear Combinations . . . . .	19
1.1.4	Limit Direction of a Holomorphic Path . . . . .	20
1.1.5	Radial Paths . . . . .	21
1.2	Plücker Coordinates . . . . .	22
1.2.1	Local Coordinates . . . . .	22
1.2.2	The Plücker Embedding . . . . .	23
1.2.3	Lines in $\mathbb{P}_3$ . . . . .	24
1.2.4	The Plücker Image . . . . .	25
1.2.5	Plücker Relations . . . . .	27
1.2.6	Systems of Vector Valued Functions . . . . .	29
1.3	Incidences and Duality . . . . .	31
1.3.1	Equations and Generators in Terms of Plücker Coordinates . . . . .	31
1.3.2	Flag Varieties . . . . .	32
1.3.3	Duality of Grassmannians . . . . .	33
1.3.4	Dual Projective Spaces . . . . .	33

1.4	Tangents to Grassmannians . . . . .	34
1.4.1	Tangents to Projective Space . . . . .	34
1.4.2	The Tangent Space of the Grassmannian . . . . .	35
1.5	Curves in Grassmannians . . . . .	37
1.5.1	The Drill . . . . .	37
1.5.2	Derived Curves . . . . .	39
1.5.3	Sums and Intersections . . . . .	43
1.5.4	Associated Curves and Curves with Prescribed Drill . . . . .	45
1.5.5	Normal Form . . . . .	47
<b>2</b>	<b>Ruled Varieties</b>	<b>49</b>
2.1	Incidence Varieties and Duality . . . . .	49
2.1.1	Unions of Linear Varieties . . . . .	49
2.1.2	Fano Varieties . . . . .	50
2.1.3	Joins . . . . .	51
2.1.4	Conormal Bundle and Dual Variety . . . . .	52
2.1.5	Duality Theorem . . . . .	55
2.1.6	The Contact Locus . . . . .	56
2.1.7	The Dual Curve . . . . .	57
2.1.8	Rational Curves . . . . .	59
2.2	Developable Varieties . . . . .	61
2.2.1	Rulings . . . . .	61
2.2.2	Adapted Parameterizations . . . . .	63
2.2.3	Germes of Rulings . . . . .	64
2.2.4	Developable Rulings and Focal Points . . . . .	65
2.2.5	Developability of Joins . . . . .	69
2.2.6	Dual Varieties of Cones and Degenerate Varieties . . . . .	71
2.2.7	Tangent and Osculating Scrolls . . . . .	74
2.2.8	Classification of Developable One Parameter Rulings . . . . .	77

---

2.2.9	Example of a “Twisted Plane” . . . . .	78
2.2.10	Characterization of Drill One Curves . . . . .	82
2.3	The Gauß Map . . . . .	85
2.3.1	Definition of the Gauß Map . . . . .	85
2.3.2	Linearity of the Fibers . . . . .	86
2.3.3	Gauß Map and Developability . . . . .	88
2.3.4	Gauß Image and Dual Variety . . . . .	88
2.3.5	Existence of Varieties with Given Gauß Rank . . . . .	89
2.4	The Second Fundamental Form . . . . .	93
2.4.1	Definition of the Second Fundamental Form . . . . .	93
2.4.2	The Degeneracy Space . . . . .	96
2.4.3	The Degeneracy Map . . . . .	97
2.4.4	The Singular and Base Locus . . . . .	98
2.4.5	The Codimension of a Uniruled Variety . . . . .	99
2.4.6	Fibers of the Gauß Map . . . . .	101
2.4.7	Characterization of Gauß Images . . . . .	103
2.4.8	Singularities of the Gauß Map . . . . .	106
2.5	Gauß Defect and Dual Defect . . . . .	109
2.5.1	Dual Defect of Segre Varieties . . . . .	110
2.5.2	Gauß Defect and Singular Locus . . . . .	111
2.5.3	Dual Defect and Singular Locus . . . . .	112
2.5.4	Computation of the Dual Defect . . . . .	113
2.5.5	The Surface Case . . . . .	115
2.5.6	Classification of Developable Hypersurfaces . . . . .	116
2.5.7	Dual Defect of Uniruled Varieties . . . . .	117
2.5.8	Varieties with Very Small Dual Varieties . . . . .	118

<b>3</b>	<b>Tangent and Secant Varieties</b>	<b>119</b>
3.1	Zak's Theorems . . . . .	119
3.1.1	Tangent Spaces, Tangent Cones, and Tangent Stars . . . . .	119
3.1.2	Zak's Theorem on Tangent and Secant Varieties . . . . .	121
3.1.3	Theorem on Tangencies . . . . .	124
3.2	Third and Higher Fundamental Forms . . . . .	125
3.2.1	Definition . . . . .	125
3.2.2	Vanishing of Fundamental Forms . . . . .	128
3.3	Tangent Varieties . . . . .	129
3.3.1	The Dimension of the Tangent Variety . . . . .	129
3.3.2	Developability of the Tangent Variety . . . . .	130
3.3.3	Singularities of the Tangent Variety . . . . .	133
3.4	The Dimension of the Secant Variety . . . . .	135
	<b>Bibliography</b>	<b>137</b>
	<b>Index</b>	<b>140</b>
	<b>List of Symbols</b>	<b>142</b>