

Contents

1. Introduction	1
1.1 Brownian Motion	1
1.1.1 Deterministic Differential Equation	1
1.1.2 Stochastic Differential Equation	2
1.1.3 Equation of Motion for the Distribution Function	3
1.2 Fokker-Planck Equation	4
1.2.1 Fokker-Planck Equation for One Variable	4
1.2.2 Fokker-Planck Equation for N Variables	5
1.2.3 How Does a Fokker-Planck Equation Arise?	5
1.2.4 Purpose of the Fokker-Planck Equation	6
1.2.5 Solutions of the Fokker-Planck Equation	7
1.2.6 Kramers and Smoluchowski Equations	7
1.2.7 Generalizations of the Fokker-Planck Equation	8
1.3 Boltzmann Equation	9
1.4 Master Equation	11
2. Probability Theory	13
2.1 Random Variable and Probability Density	13
2.2 Characteristic Function and Cumulants	16
2.3 Generalization to Several Random Variables	19
2.3.1 Conditional Probability Density	21
2.3.2 Cross Correlation	21
2.3.3 Gaussian Distribution	23
2.4 Time-Dependent Random Variables	25
2.4.1 Classification of Stochastic Processes	26
2.4.2 Chapman-Kolmogorov Equation	28
2.4.3 Wiener-Khintchine Theorem	29
2.5 Several Time-Dependent Random Variables	30
3. Langevin Equations	32
3.1 Langevin Equation for Brownian Motion	32
3.1.1 Mean-Squared Displacement	34
3.1.2 Three-Dimensional Case	36
3.1.3 Calculation of the Stationary Velocity Distribution Function	36

3.2 Ornstein-Uhlenbeck Process	38
3.2.1 Calculation of Moments	39
3.2.2 Correlation Function	41
3.2.3 Solution by Fourier Transformation	42
3.3 Nonlinear Langevin Equation, One Variable	44
3.3.1 Example	45
3.3.2 Kramers-Moyal Expansion Coefficients	48
3.3.3 Itô's and Stratonovich's Definitions	50
3.4 Nonlinear Langevin Equations, Several Variables	54
3.4.1 Determination of the Langevin Equation from Drift and Diffusion Coefficients	56
3.4.2 Transformation of Variables	57
3.4.3 How to Obtain Drift and Diffusion Coefficients for Systems	58
3.5 Markov Property	59
3.6 Solutions of the Langevin Equation by Computer Simulation	60
4. Fokker-Planck Equation	63
4.1 Kramers-Moyal Forward Expansion	63
4.1.1 Formal Solution	66
4.2 Kramers-Moyal Backward Expansion	67
4.2.1 Formal Solution	69
4.2.2 Equivalence of the Solutions of the Forward and Backward Equations	69
4.3 Pawula Theorem	70
4.4 Fokker-Planck Equation for One Variable	72
4.4.1 Transition Probability Density for Small Times	73
4.4.2 Path Integral Solutions	74
4.5 Generation and Recombination Processes	76
4.6 Application of Truncated Kramers-Moyal Expansions	77
4.7 Fokker-Planck Equation for N Variables	81
4.7.1 Probability Current	84
4.7.2 Joint Probability Distribution	85
4.7.3 Transition Probability Density for Small Times	85
4.8 Examples for Fokker-Planck Equations with Several Variables	86
4.8.1 Three-Dimensional Brownian Motion without Position Variable	86
4.8.2 One-Dimensional Brownian Motion in a Potential	87
4.8.3 Three-Dimensional Brownian Motion in an External Force	87
4.8.4 Brownian Motion of Two Interacting Particles in an External Potential	88
4.9 Transformation of Variables	88
4.10 Covariant Form of the Fokker-Planck Equation	91
5. Fokker-Planck Equation for One Variable; Methods of Solution	96
5.1 Normalization	96
5.2 Stationary Solution	98

5.3	Ornstein-Uhlenbeck Process	99
5.4	Eigenfunction Expansion	101
5.5	Examples	108
5.5.1	Parabolic Potential	108
5.5.2	Inverted Parabolic Potential	109
5.5.3	Infinite Square Well for the Schrödinger Potential	110
5.5.4	<i>V</i> -Shaped Potential for the Fokker-Planck Equation	111
5.6	Jump Conditions	112
5.7	A Bistable Model Potential	114
5.8	Eigenfunctions and Eigenvalues of Inverted Potentials	117
5.9	Approximate and Numerical Methods for Determining Eigenvalues and Eigenfunctions	119
5.9.1	Variational Method	120
5.9.2	Numerical Integration	120
5.9.3	Expansion into a Complete Set	121
5.10	Diffusion Over a Barrier	122
5.10.1	Kramers' Escape Rate	123
5.10.2	Bistable and Metastable Potential	125
6.	Fokker-Planck Equation for Several Variables; Methods of Solution ..	133
6.1	Approach of the Solutions to a Limit Solution	134
6.2	Expansion into a Biorthogonal Set	137
6.3	Transformation of the Fokker-Planck Operator, Eigenfunction Expansions	139
6.4	Detailed Balance	145
6.5	Ornstein-Uhlenbeck Process	153
6.6	Further Methods for Solving the Fokker-Planck Equation	158
6.6.1	Transformation of Variables	158
6.6.2	Variational Method	158
6.6.3	Reduction to an Hermitian Problem	159
6.6.4	Numerical Integration	159
6.6.5	Expansion into a Complete Set	159
6.6.6	Matrix Continued-Fraction Method	160
6.6.7	WKB Method	162
7.	Linear Response and Correlation Functions	163
7.1	Linear Response Function	164
7.2	Correlation Functions	166
7.3	Susceptibility	172
8.	Reduction of the Number of Variables	179
8.1	First-Passage Time Problems	179
8.2	Drift and Diffusion Coefficients Independent of Some Variables	183
8.2.1	Time Integrals of Markovian Variables	184

8.3 Adiabatic Elimination of Fast Variables	188
8.3.1 Linear Process with Respect to the Fast Variable	192
8.3.2 Connection to the Nakajima-Zwanzig Projector Formalism	194
9. Solutions of Tridiagonal Recurrence Relations, Application to Ordinary and Partial Differential Equations	196
9.1 Applications and Forms of Tridiagonal Recurrence Relations ...	197
9.1.1 Scalar Recurrence Relation	197
9.1.2 Vector Recurrence Relation	199
9.2 Solutions of Scalar Recurrence Relations	203
9.2.1 Stationary Solution	203
9.2.2 Initial Value Problem	209
9.2.3 Eigenvalue Problem	214
9.3 Solutions of Vector Recurrence Relations	216
9.3.1 Initial Value Problem	217
9.3.2 Eigenvalue Problem	220
9.4 Ordinary and Partial Differential Equations with Multiplicative Harmonic Time-Dependent Parameters	222
9.4.1 Ordinary Differential Equations	222
9.4.2 Partial Differential Equations	225
9.5 Methods for Calculating Continued Fractions	226
9.5.1 Ordinary Continued Fractions	226
9.5.2 Matrix Continued Fractions	227
10. Solutions of the Kramers Equation	229
10.1 Forms of the Kramers Equation	229
10.1.1 Normalization of Variables	230
10.1.2 Reversible and Irreversible Operators	231
10.1.3 Transformation of the Operators	233
10.1.4 Expansion into Hermite Functions	234
10.2 Solutions for a Linear Force	237
10.2.1 Transition Probability	238
10.2.2 Eigenvalues and Eigenfunctions	241
10.3 Matrix Continued-Fraction Solutions of the Kramers Equation ..	249
10.3.1 Initial Value Problem	251
10.3.2 Eigenvalue Problem	255
10.4 Inverse Friction Expansion	257
10.4.1 Inverse Friction Expansion for $K_0(t)$, $G_{0,0}(t)$ and $L_0(t)$..	259
10.4.2 Determination of Eigenvalues and Eigenvectors	266
10.4.3 Expansion for the Green's Function $G_{n,m}(t)$	268
10.4.4 Position-Dependent Friction	275
11. Brownian Motion in Periodic Potentials	276
11.1 Applications	280
11.1.1 Pendulum	280

11.1.2 Superionic Conductor	280
11.1.3 Josephson Tunneling Junction	281
11.1.4 Rotation of Dipoles in a Constant Field	282
11.1.5 Phase-Locked Loop	283
11.1.6 Connection to the Sine-Gordon Equation	285
11.2 Normalization of the Langevin and Fokker-Planck Equations ..	286
11.3 High-Friction Limit	287
11.3.1 Stationary Solution	287
11.3.2 Time-Dependent Solution	294
11.4 Low-Friction Limit	300
11.4.1 Transformation to E and x Variables	301
11.4.2 ‘Ansatz’ for the Stationary Distribution Functions	304
11.4.3 x -Independent Functions	306
11.4.4 x -Dependent Functions	307
11.4.5 Corrected x -Independent Functions and Mobility	310
11.5 Stationary Solutions for Arbitrary Friction	314
11.5.1 Periodicity of the Stationary Distribution Function	315
11.5.2 Matrix Continued-Fraction Method	317
11.5.3 Calculation of the Stationary Distribution Function	320
11.5.4 Alternative Matrix Continued Fraction for the Cosine Potential	325
11.6 Bistability between Running and Locked Solution	328
11.6.1 Solutions Without Noise	329
11.6.2 Solutions With Noise	334
11.6.3 Low-Friction Mobility With Noise	335
11.7 Instationary Solutions	337
11.7.1 Diffusion Constant	342
11.7.2 Transition Probability for Large Times	343
11.8 Susceptibilities	347
11.8.1 Zero-Friction Limit	355
11.9 Eigenvalues and Eigenfunctions	359
11.9.1 Eigenvalues and Eigenfunctions in the Low-Friction Limit	365
12. Statistical Properties of Laser Light	374
12.1 Semiclassical Laser Equations	377
12.1.1 Equations Without Noise	377
12.1.2 Langevin Equation	379
12.1.3 Laser Fokker-Planck Equation	382
12.2 Stationary Solution and Its Expectation Values	384
12.3 Expansion in Eigenmodes	387
12.4 Expansion into a Complete Set; Solution by Matrix Continued Fractions	394
12.4.1 Determination of Eigenvalues	396
12.5 Transient Solution	398
12.5.1 Eigenfunction Method	398
12.5.2 Expansion into a Complete Set	401
12.5.3 Solution for Large Pump Parameters	404

12.6 Photoelectron Counting Distribution	408
12.6.1 Counting Distribution for Short Intervals	409
12.6.2 Expectation Values for Arbitrary Intervals	412
Appendices	414
A1 Stochastic Differential Equations with Colored Gaussian Noise	414
A2 Boltzmann Equation with BGK and SW Collision Operators ...	420
A3 Evaluation of a Matrix Continued Fraction for the Harmonic Oscillator	422
A4 Damped Quantum-Mechanical Harmonic Oscillator	425
A5 Alternative Derivation of the Fokker-Planck Equation	429
A6 Fluctuating Control Parameter	431
References	436
Subject Index	445