

CONTENTS

CONTENTS OF VOLUME II	xi
TRANSLATOR'S PREFACE	xvii
AUTHOR'S PREFACE	xix
NOTATION	xxiii
INTRODUCTORY MATERIAL	1
I. ANALYTICAL THEOREMS	1
1. Abel's transformation	1
2. Second mean value theorem	3
3. Convex curves and convex sequences	3
II. NUMERICAL SERIES, SUMMATION	5
4. Series with monotonically decreasing terms	5
5. Linear methods of summation	9
6. Method of arithmetic means [or $(C, 1)$]	11
7. Abel's method	12
III. INEQUALITIES FOR NUMBERS, SERIES AND INTEGRALS	16
8. Numerical inequalities	16
9. Hölder's inequality	18
10. Minkowski's inequality	21
11. O - and o -relationships for series and integrals	22
IV. THEORY OF SETS AND THEORY OF FUNCTIONS	25
12. On the upper limit of a sequence of sets	25
13. Convergence in measure	25
14. Passage to the limit under Lebesgue's integral sign	26
15. Lebesgue points	27
16. Riemann-Stieltjes integral	29
17. Helly's two theorems	30
18. Fubini's theorem	31
V. FUNCTIONAL ANALYSIS	31
19. Linear functionals in C	31
20. Linear functionals in L^p ($p > 1$)	32
21. Convergence in norm in the spaces L^p	33

VI. THEORY OF APPROXIMATION OF FUNCTIONS BY TRIGONOMETRIC POLYNOMIALS	34
22. Elementary properties of trigonometric polynomials	34
23. Bernstein's inequality	35
24. Trigonometric polynomial of best approximation	36
25. Modulus of continuity, modulus of smoothness, and integral modulus of continuity	37
CHAPTER I. BASIC CONCEPTS AND THEOREMS IN THE THEORY OF TRIGONOMETRIC SERIES	43
1. The concept of a trigonometric series; conjugate series	43
2. The complex form of a trigonometric series	44
3. A brief historical synopsis	45
4. Fourier formulae	46
5. The complex form of a Fourier series	47
6. Problems in the theory of Fourier series; Fourier–Lebesgue series	48
7. Expansion into a trigonometric series of a function with period $2l$	49
8. Fourier series for even and odd functions	50
9. Fourier series with respect to the orthogonal system	51
10. Completeness of an orthogonal system	54
11. Completeness of the trigonometric system in the space L	55
12. Uniformly convergent Fourier series	58
13. The minimum property of the partial sums of a Fourier series; Bessel's inequality	59
14. Convergence of a Fourier series in the metric space L^2	60
15. Concept of the closure of the system. Relationship between closure and completeness	61
16. The Riesz–Fischer theorem	64
17. The Riesz–Fischer theorem and Parseval's equality for a trigonometric system	64
18. Parseval's equality for the product of two functions	65
19. The tending to zero of Fourier coefficients	66
20. Fejér's lemma	67
21. Estimate of Fourier coefficients in terms of the integral modulus of continuity of the function	70
22. Fourier coefficients for functions of bounded variation	71
23. Formal operations on Fourier series	72
24. Fourier series for repeatedly differentiated functions	79
25. On Fourier coefficients for analytic functions	80
26. The simplest cases of absolute and uniform convergence of Fourier series	83
27. Weierstrass's theorem on the approximation of a continuous function by trigonometric polynomials	84
28. The density of a class of trigonometric polynomials in the spaces L^p ($p \geq 1$)	85
29. Dirichlet's kernel and its conjugate kernel	85
30. Sine or cosine series with monotonically decreasing coefficients	87
31. Integral expressions for the partial sums of a Fourier series and its conjugate series	95
32. Simplification of expressions for $S_n(x)$ and $\bar{S}_n(x)$	100
33. Riemann's principle of localization	103
34. Steinhaus's theorem	104

35. Integral $\int_0^{\infty} [(\sin x)/x] dx$. Lebesgue constants	105
36. Estimate of the partial sums of a Fourier series of a bounded function	110
37. Criterion of convergence of a Fourier series	111
38. Dini's test	113
39. Jordan's test	114
40. Integration of Fourier series	116
41. Gibbs's phenomenon	117
42. Determination of the magnitude of the discontinuity of a function from its Fourier series	121
43. Singularities of Fourier series of continuous functions. Fejér polynomials	123
44. A continuous function with a Fourier series which converges everywhere but not uniformly	125
45. Continuous function with a Fourier series divergent at one point (Fejér's example)	127
46. Divergence at one point (Lebesgue's example)	128
47. Summation of a Fourier series by Fejér's method	133
48. Corollaries of Fejér's theorem	137
49. Fejér-Lebesgue theorem	139
50. Estimate of the partial sums of a Fourier series	141
51. Convergence factors	143
52. Comparison of Dirichlet and Fejér kernels	143
53. Summation of Fourier series by the Abel-Poisson method	149
54. Poisson kernel and Poisson integral	150
55. Behaviour of the Poisson integral at points of continuity of a function	152
56. Behaviour of a Poisson integral in the general case	154
57. The Dirichlet problem	159
58. Summation by Poisson's method of a differentiated Fourier series	160
59. Poisson-Stieltjes integral	162
60. Fejér and Poisson sums for different classes of functions	164
61. General trigonometric series. The Lusin-Denjoy theorem	173
62. The Cantor-Lebesgue theorem	174
63. An example of an everywhere divergent series with coefficients tending to zero	176
64. A study of the convergence of one class of trigonometric series	177
65. Lacunary sequences and lacunary series	178
66. Smooth functions	181
67. The Schwarz second derivative	186
68. Riemann's method of summation	189
69. Application of Riemann's method of summation to Fourier series	192
70. Cantor's theorem of uniqueness	193
71. Riemann's principle of localization for general trigonometric series	195
72. du Bois-Reymond's theorem	201
73. Problems	204
CHAPTER II. FOURIER COEFFICIENTS	209
1. Introduction	209
2. The order of Fourier coefficients for functions of bounded variation. Criterion for the continuity of functions of bounded variation	210
3. Concerning Fourier coefficients for functions of the class $\text{Lip } \alpha$	215
4. The relationship between the order of summability of a function and the Fourier coefficients	217

5. The generalization of Parseval's equality for the product of two functions	225
6. The rate at which the Fourier coefficients of summable functions tend to zero	228
7. Auxiliary theorems concerning the Rademacher system	230
8. Absence of criteria applicable to the moduli of coefficients	233
9. Some necessity conditions for Fourier coefficients	236
10. Salem's necessary and sufficient conditions	239
11. The trigonometric problem of moments	242
12. Coefficients of trigonometric series with non-negative partial sums	244
13. Transformation of Fourier series	252
14. Problems	254
CHAPTER III. THE CONVERGENCE OF A FOURIER SERIES AT A POINT	260
1. Introduction	260
2. Comparison of the Dini and Jordan tests	260
3. The de la Vallée-Poussin test and its comparison with the Dini and Jordan tests	261
4. The Young test	263
5. The relationship between the Young test and the Dini, Jordan and de la Vallée-Poussin tests	266
6. The Lebesgue test	269
7. A comparison of the Lebesgue test with all the preceding tests	274
8. The Lebesgue-Gergen test	279
9. Concerning the necessity conditions for convergence at a point	285
10. Sufficiency convergence tests at a point with additional restrictions on the coefficients of the series	289
11. A note concerning the uniform convergence of a Fourier series in some interval	292
12. Problems	293
CHAPTER IV. FOURIER SERIES OF CONTINUOUS FUNCTIONS	296
1. Introduction	296
2. Sufficiency conditions for uniform convergence, expressed in terms of Fourier coefficients	297
3. Sufficiency conditions for uniform convergence in terms of the best approximations	300
4. The Dini-Lipschitz test	301
5. The Salem test. Functions of Φ -bounded variation	305
6. The Rogosinski identity	310
7. A test of uniform convergence, using the integrated series	314
8. The generalization of the Dini-Lipschitz test (in the integral form)	315
9. Uniform convergence over the interval $[a, b]$	319
10. The Sato test	322
11. Concerning uniform convergence near every point of an interval	326
12. Concerning operations on functions to obtain uniformly convergent Fourier series	327
13. Concerning uniform convergence by rearrangement of the signs in the terms of the series	330
14. Extremal properties of some trigonometric polynomials	332
15. The choice of arguments for given moduli of the terms of the series	334

16. Concerning Fourier coefficients of continuous functions	336
17. Concerning the singularities of Fourier series of continuous functions	342
18. A continuous function with a Fourier series non-uniformly convergent in any interval	342
19. Concerning a set of points of divergence for a trigonometric series	344
20. A continuous function with a Fourier series divergent in a set of the power of the continuum	345
21. Divergence in a given denumerable set	346
22. Divergence in a set of the power of the continuum for bounded partial sums	348
23. Divergence for a series of $f^2(x)$	350
24. Sub-sequences of partial sums of Fourier series for continuous functions	354
25. Resolution into the sum of two series convergent in sets of positive measure	357
26. Problems	358

CHAPTER V. CONVERGENCE AND DIVERGENCE OF A FOURIER SERIES IN A SET

	362
1. Introduction	362
2. The Kolmogorov–Seliverstov and Plessner theorem	363
3. A convergence test expressed by the first differences of the coefficients	369
4. Convergence factors	370
5. Other forms of the condition imposed in the Kolmogorov–Seliverstov and Plessner theorem	371
6. Corollaries of Plessner's theorem	373
7. Concerning the equivalence of some conditions expressed in terms of integrals and in terms of series	375
8. A test of almost everywhere convergence for functions of $L^p(1 \leq p \leq 2)$	379
9. Expression of the conditions of almost everywhere convergence in terms of the quadratic moduli of continuity and the best approximations	381
10. Tests of almost everywhere convergence in an interval of length less than 2π	384
11. Indices of convergence	389
12. The convex capacity of sets	398
13. A convergence test, using an integrated series	414
14. The Salem test	416
15. The Marcinkiewicz test	417
16. Convergence test expressed by the logarithmic measure of the set	421
17. Fourier series, almost everywhere divergent	430
18. The impossibility of strengthening the Marcinkiewicz test	443
19. Concerning the series conjugate to an almost everywhere divergent Fourier series	447
20. A Fourier series, divergent at every point	455
21. Concerning the principle of localization for sets	465
22. Concerning the convergence of a Fourier series in a given set and its divergence outside it	470
23. The problem of convergence and the principle of localization for Fourier series with rearranged terms	480
24. Problems	483

CHAPTER VI. "ADJUSTMENT" OF FUNCTIONS IN A SET OF SMALL MEASURE	488
1. Introduction	488
2. Two elementary lemmas	488
3. Lemma concerning the Dirichlet factor	490
4. "Adjustment" of a function to obtain a uniformly convergent Fourier series	500
5. The strengthened C -property	510
6. Problems connected with the "adjustment" of functions	511
7. "Adjustment" of a summable function outside a given perfect set	512
8. Problems	526
APPENDIX	
TO CHAPTER II	528
1. The Phragmén–Lindelöf principle	528
2. Modulus of continuity and modulus of smoothness in $L^p(p \geq 1)$	528
3. A converse of the Hölder inequality	529
4. The Banach–Steinhaus theorem	531
TO CHAPTER IV	532
5. Categories of sets	532
6. Riemann's and Carathéodory's theorems	532
7. The connection between the modulus of continuity and the best approximation of a function	533
TO CHAPTER V	536
8. μ -measures and integrals	536
BIBLIOGRAPHY	537
INDEX	549