Contents

Preface v

1 Introduction 1

2 Review of Ordinary Differential Equations 5
2.1 First Order Linear Differential Equations 6
2.1.1 First Order Constant Coefficient Linear Differential Equations 7
2.1.2 Variable Coefficient First Order Linear Differential Equations 9
2.1.3 Equations Reducible to Linear Differential Equations 11
2.1.4 Qualitative Solution: Phase Diagrams 12
2.1.5 Some Economic Applications 14
1. Walrasian Tâtonnement Process 14
2. The Keynesian Model 15
3. Harrod Domar's Economic Growth Model 16
4. Domar's Debt Model (1944) 17
5. Profit and Investment 17
6. The Neo-Classical Model of Economic Growth 18
2.2 Second and Higher Order Linear Differential Equations 19
2.2.1 Particular Integral (x_p or x_e) where $d(t) = d$ Constants 24
2.2.2 Particular Integral (x_p) when $d = g(t)$ is some Function of t . 25
1. The Undetermined Coefficients Method 25
2. Inverse Operator Method 26
3. Laplace Transform Method 29
2.3 Higher Order Linear Differential Equations with Constant Coefficients 31
2.4 Stability Conditions 33
2.5 Some Economic Applications 34
1. The IS-LM Model of the Economy 34
2. A Continuous Multiplier-Acceleration Model 35
3. Stabilization Policies 35
4. Equilibrium Models with Stock 37
2.6 Conclusion 38

3 Review of Difference Equations 39
3.1 Introduction 39
3.2 First Order Difference Equations 40
3.2.1 Linear Difference Equations 40

3.2.2 Non-linear Difference Equations and Phase Diagram 42
3.2.3 Some Economic Applications 43
1. The Cobweb Cycle 43
2. The Dynamic Multiplier Model 44
3. The Overlapping Generations Model 44
3.3 Second Order Linear Difference Equations 46
3.3.1 Particular Integral 46
3.3.2 The Complementary Functions $x_c(t)$ 47
3.3.3 Complete Solution and Examples 49
3.4 Higher Order Difference Equations 51
3.5 Stability Conditions 51
3.5.1 Stability of First Order Difference Equations 52
3.5.2 Stability of Second Order Difference Equations 52
3.5.3 Stability of Higher Order Difference Equations 53
3.6 Economic Applications 54
3.6.1 Samuelson's (1939) Business Cycle 54
3.6.2 Hick's (1950) Contribution to the Theory of Trade Cycle . . . 55
3.7 Concluding Remarks 57

4 Review of Some Linear Algebra 59
4.1 Vector and Vector Spaces 59
4.1.1 Vector Spaces 60
4.1.2 Inner Product Space 61
4.1.3 Null Space and Range, Rank and Kernel 62
4.2 Matrices 63
4.2.1 Some Special Matrices 63
4.2.2 Matrix Operations 64
4.3 Determinant Functions 65
4.3.1 Properties of Determinants 65
4.3.2 Computations of Determinants 66
4.4 Matrix Inversion and Applications 68
4.5 Eigenvalues and Eigenvectors 69
4.5.1 Similar Matrices 72
4.5.2 Real Symmetric Matrices 73
4.6 Quadratic Forms 74
4.7 Diagonalization of Matrices 76
4.7.1 Real Eigenvalues 76
4.7.2 Complex Eigenvalues and Eigenvectors 78
4.8 Jordan Canonical Form 79
4.9 Idempotent Matrices and Projection 81
4.10 Conclusion 82

5 First Order Differential Equations Systems 83
5.1 Introduction . . . 83
5.2 Constant Coefficient Linear Differential Equation (ODE) Systems . . 83
5.2.1 Case (i). Real and Distinct Eigenvalues . . . 84
5.2.2 Case (ii). Repeated Eigenvalues . . . 87
5.2.3 Case (iii). Complex Eigenvalues . . . 88
5.3 Jordan Canonical Form of ODE Systems . . . 89
Case (i) Real Distinct Eigenvalues . . . 90
Case (ii) Multiple Eigenvalues . . . 91
Case (iii) Complex Eigenvalues . . . 93
5.4 Alternative Methods for Solving $\dot{x} = Ax$ . . . 95
5.4.1 Sylvester's Method . . . 95
5.4.2 Putzer's Methods (Putzer 1966) . . . 96
5.4.3 A Direct Method of Solving $\dot{x} = Ax$ . . . 97
5.5 Reduction to First Order of ODE Systems . . . 98
5.6 Fundamental Matrix . . . 98
5.7 Stability Conditions of ODE Systems . . . 100
5.7.1 Asymptotic Stability . . . 100
5.7.2 Global Stability: Liapunov's Second Method . . . 101
5.8 Qualitative Solution: Phase Portrait Diagrams . . . 102
5.9 Some Economic Applications . . . 107
5.9.1 Dynamic IS-LM Keynesian Model . . . 107
5.9.2 Dynamic Leontief Input-Output Model . . . 109
5.9.3 Multimarket Equilibrium . . . 111
5.9.4 Walras-Cassel-Leontief General Equilibrium Model . . . 112

6 First Order Difference Equations Systems 115
6.1 First Order Linear Systems . . . 115
6.2 Jordan Canonical Form . . . 117
Case (i). Real Distinct Eigenvalues . . . 118
Case (ii). Multiple Eigenvalues . . . 119
Case (iii). Complex Eigenvalues . . . 120
6.3 Reduction to First Order Systems . . . 121
6.4 Stability Conditions . . . 123
6.4.1 Local Stability . . . 123
6.4.2 Global Stability . . . 125
6.5 Qualitative Solutions: Phase Diagrams . . . 126
6.6 Some Economic Applications . . . 128
1. A Multisectoral Multiplier-Accelerator Model . . . 128
2. Capital Stock Adjustment Model . . . 129
3. Distributed Lags Model . . . 129
4. Dynamic Input-Output Model . . . 130

7 Nonlinear Systems 133
7.1 Introduction . . . 133
7.2 Linearization Theory . . . 134
7.2.1 Linearization of Dynamic Systems in the Plane . . . 136
7.2.2 Linearization Theory in Three Dimensions . . . 144
7.2.3 Linearization Theory in Higher Dimensions . . . 145
7.3 Qualitative Solution: Phase Diagrams . . . 147
7.4 Limit Cycles . . . 149
Economic Application I: Kaldor's Trade Cycle Model . . . 152
7.5 The Liénard-Van der Pol Equations and the Uniqueness of Limit Cycles . . . 154
Economic Application II: Kaldor's Model as a Liénard Equation . . . 156
7.6 Linear and Nonlinear Maps . . . 157
7.7 Stability of Dynamical Systems . . . 159
7.7.1 Asymptotic Stability . . . 159
7.7.2 Structural Stability . . . 160
7.8 Conclusion . . . 161

8 Gradient Systems, Lagrangean and Hamiltonian Systems 163
8.1 Introduction . . . 163
8.2 The Gradient Dynamic Systems (GDS) . . . 163
8.3 Lagrangean and Hamiltonian Systems . . . 167
8.4 Hamiltonian Dynamics . . . 170
8.4.1 Conservative Hamiltonian Dynamic Systems (CHDS) . . . 171
8.4.2 Perturbed Hamiltonian Dynamic Systems (PHDS) . . . 174
8.5 Economic Applications . . . 176
8.5.1 Hamiltonian Dynamic Systems (HDS) in Economics . . . 176
8.5.2 Gradient (GDS) vs Hamiltonian (HDS) Systems in Economics 177
8.5.3 Economic Applications: Two-State-Variables Optimal Economic Control Models . . . 178
8.6 Conclusion . . . 181

9 Simplifying Dynamical Systems 183
9.1 Introduction . . . 183
9.2 Poincaré Map . . . 183
9.3 Floquet Theory . . . 185
9.4 Centre Manifold Theorem (CMT) . . . 187
9.5 Normal Forms . . . 191
9.6 Elimination of Passive Coordinates . . . 192
9.7 Liapunov–Schmidt Reduction . . . 193
9.8 Economic Applications and Conclusions . . . 194

10 Bifurcation, Chaos and Catastrophes in Dynamical Systems 195
10.1 Introduction . . . 195
10.2 Bifurcation Theory (BT) . . . 195

10.2.1 One Dimensional Bifurcations 197
10.2.2 Hopf Bifurcation 200
10.2.3 Some Economic Applications 204
1. The Keynesian IS-LM Model 204
2. Hopf Bifurcation in an Advertising Model 205
3. A Dynamic Demand Supply Model 207
4. Generalized Tobin's Model of Money and Economic Growth 208
10.2.4 Bifurcations in Discrete Dynamical Systems 209
1. The Fold of Saddle Node Bifurcation 209
2. Transcritical Bifurcation 210
3. Flip Bifurcation 210
4. Logistic System 210
10.3 Chaotic or Complex Dynamical Systems (DS) 211
10.3.1 Chaos in Unimodal Maps in Discrete Systems 212
10.3.2 Chaos in Higher Dimensional Discrete Systems 216
10.3.3 Chaos in Continuous Systems 216
10.3.4 Routes to Chaos 217
1. Period Doubling and Intermittency 217
2. Horseshoe and Homoclinic Orbits 218
10.3.5 Liapunov Characteristic Exponent (LCE) and Attractor's Dimension 221
10.3.6 Some Economic Applications 222
1. Chaotic Dynamics in a Macroeconomic Model 222
2. Erratic Demand of the Rich 224
3. Structure and Stability of Inventory Cycles 224
4. Chaotic economic Growth with Pollution 225
5. Chaos in Business Cycles 226
10.4 Catastrophe Theory (C.T.) 226
10.4.1 Some General Concepts 227
10.4.2 The Morse and Splitting Lemma 228
10.4.3 Codimension and Unfolding 229
10.4.4 Classification of Singularities 231
10.4.5 Some Elementary Catastrophes 232
1. The Fold Catastrophe 232
2. The Cusp Catastrophe 233
10.4.6 Some Economic Applications 235
1. The Shutdown of the Firm (Tu 1982) 235
2. Kaldor's Trade Cycle 236
3. A Catastrophe Theory of Defence Expenditure 238
4. Innovation, Industrial Evolution and Revolution 240
10.4.7 Comparative Statics (C.S.), Singularities and Unfolding 241
10.5 Concluding Remarks 243

11 Optimal Dynamical Systems 245
11.1 Introduction . . . 245
11.2 Pontryagin's Maximum Principle . . . 245
11.2.1 First Variations and Necessary Conditions . . . 248
11.2.2 Second Variations and Sufficient Conditions . . . 252
11.3 Stabilization Control Models . . . 253
11.4 Some Economic Applications . . . 256
1. Intergenerational Distribution of Non-renewable Resources . . 256
2. Optimal Harvesting of Renewable Resources . . . 256
3. Multiplier–Accelerator Stabilization Model . . . 257
4. Optimal Economic Growth (OEG) . . . 258
11.5 Asymptotic Stability of Optimal Dynamical Systems (ODS) . . . 260
11.6 Structural Stability of Optimal Dynamical Systems . . . 263
11.6.1 Hopf Bifurcation in Optimal Economic Control Models and Optimal Limit Cycles . . . 263
Two-State-Variable Models . . . 264
Multisectoral OEG Models . . . 265
11.6.2 Chaos in Optimal Dynamical Systems (ODS) . . . 267
11.7 Conclusion . . . 268

12 Some Applications in Economics and Biology 271
12.1 Introduction . . . 271
12.2 Economic Applications of Dynamical Systems . . . 271
12.2.1 Business Cycles Theories . . . 271
1. Linear Multiplier-Accelerator Models . . . 272
2. Nonlinear Models . . . 273
2.1. Flexible Multiplier-Accelerator Models . . . 273
2.2. Kaldor's Type of Flexible Accelerator Models . . . 275
2.3. Goodwin's Class Struggle Model . . . 275
3. Optimal Economic Fluctuations and Chaos . . . 276
12.2.2 General Equilibrium Dynamics . . . 276
Tâtonnement Adjustment Process . . . 277
Non-Tâtonnement Models . . . 278
12.2.3 Economic Growth Theories . . . 279
1. Harrod-Domar's Models . . . 279
2. Neo-Classical Models . . . 279
2.1. Two Sector Models . . . 280
2.2. Economic Growth with Money . . . 281
2.3. Optimal Economic Growth Models . . . 282
2.4. Endogenous Economic Growth Models . . . 282
12.3 Dynamical Systems in Biology . . . 284
12.3.1 One Species Growth Models . . . 284
12.3.2 Two Species Models . . . 285
1. Predation Models . . . 285
2. Competition Models . . . 288
12.3.3 The Dynamics of a Heartbeat . . . 288

12.4 Bioeconomics and Natural Resources 290
12.4.1 Optimal Management of Renewable and Exhaustible Resources 290
12.4.2 Optimal Control of Prey-Predator Models 292
(i) Control by an Ideal Pesticide 292
(ii) Biological Control 293
12.5 Conclusion 294

Bibliography 295