

Contents

Volume II. Variational Integrals

1.	Regular Variational Integrals	1
1	The Direct Methods	2
1.1	The Abstract Setting (Lower semicontinuous and coercive functionals. Weierstrass theorem. τ -relaxed functional)	3
1.2	Some Classical Lower Semicontinuity Theorems (Lower semicontinuity with respect to the weak convergence in L^1 . The role of Banach-Saks's theorem and Jensen's inequality. Regular and smooth integrands in the Calculus of Variations)	10
1.3	A General Semicontinuity Theorem (Lower semicontinuity with respect to the weak* convergence in L^1)	19
2	Polyconvex Envelops and Regular Parametric Integrals	23
2.1	Polyconvexity and Polyconvex Envelops (A few facts from convex analysis. n -vectors associated to the tangent planes to graphs. Polyconvex functions and polyconvex envelopes)	26
2.2	Parametric Polyconvex Envelops of Integrands (The parametric polyconvex l.s.c. envelop of an integrand. Mass and comass with respect to the integrand f)	37
2.3	The Parametric Extension of Regular Integrals (The parametric extension of an integral as integral of the parametric polyconvex l.s.c. envelop)	44
2.4	The Polyconvex l.s.c. Extension of Some Lagrangians (Area of graphs. The total variation of the gradient. The Dirichlet inte- gral. The p -energy functional. The liquid crystal integrand)	45
3	Regular Integrals in the Class of Cartesian Currents	74
3.1	Parametric Integrands and Lower Semicontinuity (Parametric integrands and lower semicontinuity of parametric integrals)	75
3.2	Existence of Minimizers in Classes of Cartesian Currents (Lower semicontinuity of the parametric extension of regular integrals. Existence of minimizers in subclasses of Cartesian currents)	82
3.3	Relaxed Energies in the Setting of Cartesian Currents (The relaxed functional in classes of Cartesian currents and maps)	88
3.4	Relaxed Energies in the Parametric Case (The approximation problem for parametric integrals. A theorem of Reshetnyak. Flat integrands and Federer's approximation theorem. In- teger multiplicity rectifiable and real minimizing currents)	90
4	Regular Integrals and Quasiconvexity	106
4.1	Quasiconvexity (Quasiconvexity as necessary condition for semicontinuity. Rank-one convexity and Legendre-Hadamard condition)	107
4.2	Quasiconvexity and Lower Semicontinuity (Quasiconvexity as sufficient condition for semicontinuity in classes of Sobolev maps or Cartesian currents)	117

4.3	Ellipticity and Quasiconvexity	127
	(Ellipticity, quasiconvexity and lower semicontinuity)	
5	Notes	131
2.	Finite Elasticity and Weak Diffeomorphisms	137
1	State Space and Stored Energies in Elasticity	139
1.1	Fields and Transformations	139
	(Non local and non linear structure of transformations)	
1.2	Kinematics	140
	(Bodies, states, and deformations of a body. Deformations as 3-surfaces in \mathbb{R}^6 , or as graphs of diffeomorphisms)	
1.3	Local Deformations	143
	(Infinitesimal deformations as simple tangent vectors to the deformation surface)	
1.4	Perfectly Elastic Bodies: Stored Energy, Convexity and Coercivity	147
	(Stored energy: different forms and constitutive conditions. Polyconvexity and coercivity)	
1.5	Variations and Stress	152
	(Infinitesimal variations and the notion of stress. Piola-Kirchhoff and Cauchy stress tensors. Energy-momentum tensor.)	
2	Physical Implications on Kinematics and Stored Energies	155
2.1	Kinematical Principles in Elasticity: Weak Deformations	156
	(Material body and its parts. Impenetrability of matter. Weakly invertible maps. Weak one-to-one transformations. Existence of local deformations. Weak deformations. Elastic bodies and absence of fractures. Elastic deformations)	
2.2	Frame Indifference and Isotropy	169
	(Frame indifference principle. Energies associated to isotropic materials)	
2.3	Convexity-like Conditions	170
	(Convexity is not compatible with elasticity. Noll's condition. Polyconvexity and Diff-quasiconvexity)	
2.4	Coercivity Conditions	175
	(A discussion of the coercivity conditions)	
2.5	Examples of Stored Energies	179
	(Ogden-type stored energies for isotropic materials)	
3	Weak Diffeomorphisms	182
3.1	The Classes $\text{dif}^{p,q}(\Omega, \widehat{\Omega})$	183
	(The class of (p, q) -weak diffeomorphisms. Weak convergence. Closure and compactness properties. An example of a discontinuous weak diffeomorphism)	
3.2	The Classes $\widetilde{\text{dif}}^{p,q}(\Omega, \widehat{\mathbb{R}^n})$	191
	(Weak diffeomorphisms with non-prescribed range. Closure and compactness properties. Elastic deformations as weak diffeomorphisms)	
3.3	Convergence Theorems for the Inverse Maps	199
	(Convergence of the ranges and of the inverse maps)	
3.4	General Weak Diffeomorphisms	204
	(Weak diffeomorphisms with vertical and horizontal parts: structure, closure, and compactness theorems)	
3.5	The Dif-classes	213
	(The approximation problem for weak diffeomorphisms)	
3.6	Volume Preserving Diffeomorphisms	214
	(The Jacobian determinant of weak one-to-one maps and of weak deformations)	
4	Connectivity Properties of the Range of Weak Diffeomorphisms	216

4.1	Connectivity of the Range of Sobolev Maps	217
	(Connected sets and d_{n-1} -connected sets. d_{n-1} -connected sets are "mapped" into connected sets by Sobolev maps)	
4.2	Connectivity of the Range of Weak Diffeomorphisms	219
	(Weak diffeomorphisms "map" connected sets into essentially connected sets. Weak diffeomorphisms do not produce cavitation. Examples)	
4.3	Regularity Properties of Locally Weak Invertible Maps	229
	(Weak local diffeomorphisms. Local properties of weak local diffeomorphisms. Vodopianov-Goldstein's theorem. Courant-Lebesgue lemma)	
4.4	Global Invertibility of Weak Maps	238
	(Conditions ensuring that weak local diffeomorphisms be one-to-one and homeomorphisms)	
4.5	An a.e. Open Map Theorem	243
	(A.e. open sets and a.e. continuous maps. Weak diffeomorphisms in $W^{1,p}$, $p > n - 1$, "are" a.e. open maps)	
5	Composition	247
5.1	Composition of weak deformations	248
	(Composition of one-to-one maps and of weak diffeomorphisms)	
5.2	On the Summability of Compositions	250
	(Binet's formula and the summability of the composition)	
5.3	Composition of Weak Diffeomorphisms	254
	(The action of weak diffeomorphisms on Cartesian maps and the pseudogroup structure of weak diffeomorphisms. Weak convergence of compositions)	
6	Existence of Equilibrium Configurations	260
6.1	Existence Theorems	261
	(The displacement pressure problem. Deformations with fractures)	
6.2	Equilibrium and Conservation Equations	264
	(Energy-momentum conservation law and Cauchy's equilibrium equation)	
6.3	The Cavitation Problem	268
	(Elastic deformations do not cavitate)	
7	Notes	278
3.	The Dirichlet Integral in Sobolev Spaces	281
1	Harmonic Maps Between Manifolds	281
1.1	First Variation and Inner Variations	283
	(Euler variations and Euler-Lagrange equation of the energy integral. Inner variations, energy-momentum tensor, inner and strong extremals. Conformality relations. Stationary points. Parametric minimal surfaces)	
1.2	Finding Harmonic Maps by Variational Methods	293
	(Existence and the regularity problem. Mappings from B^n into the upper hemisphere of S^n . Mappings from B^n into S^{n-1})	
2	Energy Minimizing Weak Harmonic Maps: Regularity Theory	296
2.1	Some Preliminaries. Reverse Hölder Inequalities	297
	(Some algebraic lemmas. The Dirichlet growth theorem of Morrey. Reverse Hölder inequalities with increasing supports)	
2.2	Classical Regularity Results	303
	(Morrey's regularity theorem for 2-dimensional weak harmonic maps)	
2.3	An Optimal Regularity Theorem	307
	(A partial regularity theorem and the existence and regularity of energy minimizing harmonic maps with range in a regular ball: results by Hildebrandt, Kaul, Widman, and Giaquinta, Giusti.)	
2.4	The Partial Regularity Theorem	319
	(The partial regularity theorem for energy minimizing weak harmonic maps: Schoen-Uhlenbeck result)	

3	Harmonic Maps in Homotopy Classes.....	333
3.1	The Action of $W^{1,2}$ -maps on Loops..... (Courant–Lebesgue lemma. In the two–dimensional case the action on loops is well defined for maps in $W^{1,2}$)	334
3.2	Minimizing Energy with Homotopic Constraints..... (Energy minimizing maps with prescribed action on loops. Schoen–Yau, Saks–Uhlenbeck, Lemaire, Eells–Sampson and Hamilton theorems)	336
3.3	Local Replacement by Harmonic Mappings: Bubbling.... (Jost’s replacement method and existence of minimal immersions of S^2)	337
4	Weak and Stationary Harmonic Maps with Values into S^2	339
4.1	The Partial Regularity Theory..... (An alternative proof. More on the singular set)	339
4.2	Stationary Harmonic Maps..... (Partial regularity results for stationary harmonic maps)	345
5	Notes.....	350
4.	The Dirichlet Energy for Maps into S^2	353
1	Variational Problems for Maps from a Domain of \mathbb{R}^2 into S^2	354
1.1	Harmonic Maps with Prescribed Degree..... (Homotopic equivalent maps and degree. Bubbling off of spheres. The stereographic and the modified stereographic projection. ϵ -conformal maps)	354
1.2	The Structure Theorem in $\text{cart}^{2,1}(\Omega \times S^2)$, $\Omega \subset \mathbb{R}^2$ (The structure and approximation theorems in $\text{cart}^{2,1}(\Omega \times S^2)$)	362
1.3	Existence and Regularity of Minimizers..... (The relaxed energy and existence of minimizers. Energy minimizing maps with constant boundary value: Lemaire’s theorem. The simplest chiral model and instantons. Large solution for harmonic maps: Brezis–Coron and Jost result. A global regularity result)	366
2	Variational Problems from a Domain of \mathbb{R}^3 into S^2	383
2.1	The Class $\text{cart}^{2,1}(\Omega \times S^2)$, $\Omega \subset \mathbb{R}^3$ (The D -field and homological singularities)	385
2.2	Density Results in $W^{1,2}(B^3, S^2)$ (Approximation by maps which are smooth except at a discrete set of points)	392
2.3	Dipoles and Gap Phenomenon..... (Dipoles and the approximate dipoles. Lavrentiev or gap phenomenon)	400
2.4	The Structure Theorem in $\text{cart}^{2,1}(\Omega \times S^2)$, $\Omega \subset \mathbb{R}^3$ (Structure of the vertical part of Cartesian currents in $\text{cart}^{2,1}(\Omega \times S^2)$)	409
2.5	Approximation by Smooth Graphs: Dirichlet Data..... (Weak approximation in energy by smooth graphs. The minimal connection and its continuity properties with respect to the $W^{1,2}$ -weak convergence. $\text{Cart}_\varphi^{2,1}(\bar{\Omega} \times S^2) = \text{cart}_\varphi^{2,1}(\bar{\Omega} \times S^2)$. Weak approximability by smooth maps in $W_\varphi^{1,2}(\bar{\Omega} \times S^2)$)	412
2.6	Approximation by Smooth Graphs: No Boundary Data... (T belongs to $\text{cart}^{2,1}(\Omega \times S^2)$ if and only if it can be approximated weakly and in energy by smooth graphs G_{u_k} possibly with $u_k = u_T$ on $\partial\Omega$)	419
2.7	The Dirichlet Integral in $\text{cart}^{2,1}(\Omega \times S^2)$, $\Omega \subset \mathbb{R}^3$ (The parametric polyconvex extension of the Dirichlet integral is its relaxed or Lebesgue’s extension. The relaxed of $\mathcal{D}(u, \Omega)$ in $W^{1,2}(\Omega, S^2)$)	423
2.8	Minimizers of Variational Problems..... (Variational problems and existence of minimizers)	429

2.9	A Partial Regularity Result	433
	(The absolutely continuous part u_T of minimizers T is regular except on a closed set whose Hausdorff dimension is not greater than 1. Tangent cones)	
2.10	The General Dipole Problem	449
	(The coarea formula and the minimum energy of dipoles)	
2.11	Singular Perturbations	452
	(Trying to solve Dirichlet problem by approximating by singularly perturbed functionals of the type of Ginzburg-Landau)	
3	Notes	458
5.	Some Regular and Non Regular Variational Problems	467
1	The Liquid Crystal Energy	467
1.1	The Sobolev Space Approach	470
	(Existence and regularity of equilibrium configuration)	
1.2	The Relaxed Energy	470
	(Existence of equilibrium configurations for the relaxed energy. The dipole problem. Relaxed energies in Sobolev spaces and Cartesian currents. Equilibrium configurations with fractures)	
1.3	The Dipole Problem	477
	(Approximation in energy: irrotational and solenoidal dipoles. The general dipole problem)	
2	The Dirichlet Integral in the Regular Case: Maps into S^2	485
2.1	Maps with Values in S^2	485
	(Maps from a n -dimensional space into S^2 and the class $\text{cart}^{2,1}(\Omega \times S^2)$. The $(n-2)$ - D -field)	
2.2	The Dipole Problem	489
	(Degree with respect to a $(n-3)$ -curve and the dipole problem)	
2.3	The Structure Theorem	494
	(Structure theorem for currents in $\text{cart}^{2,1}(\Omega \times S^2)$, $\Omega \subset \mathbb{R}^n$)	
3	The Dirichlet Integral in the Regular Case: Maps into a Manifold	496
3.1	The Class $\text{cart}^{2,1}(\Omega \times \mathcal{Y})$	497
	(The structure theorem for currents in $\text{cart}^{2,1}(\Omega \times \mathcal{Y})$, $\Omega \subset \mathbb{R}^n$)	
3.2	Spherical Vertical Parts and a Closure Theorem	501
	(Reduced Cartesian currents. Closure theorem. Vertical parts of currents in $\text{Cart}^{2,1}(\Omega \times \mathcal{Y})$ are of the type S^2)	
3.3	The Dirichlet Integral and Minimizers	506
4	The Dirichlet Integral in the Non Regular Case: a Homological Theory	508
4.1	(n, p) -Currents	509
	((n, p) -graphs and the classes $\mathcal{A}^{(p),1}$. rectifiable (r, p) -currents, (r, p) -mass, (r, p) -boundary. Vertical (r, p) -currents and cohomology. Integer multiplicity rectifiable vector-valued currents)	
4.2	Graphs of Sobolev Maps	516
	(Singularities of Sobolev maps and the currents $\mathbf{P}(u; \sigma)$ and $\mathbf{D}(u; \sigma)$. The class $\text{mc} - W_{\varphi}^{1,p}(\Omega, \mathcal{Y})$)	
4.3	p -Dirichlet Graphs and Cartesian Currents	525
	(The classes \mathcal{D}_p -graph $(\Omega \times \mathcal{Y})$, $\text{red-}\mathcal{D}_p$ -graph $(\Omega \times \mathcal{Y})$, and $\text{cart}^{p, \dots, 1}(\Omega \times \mathcal{Y})$: closure theorems)	
4.4	The Dirichlet Integral	534
	(Representation. Minimizers and homological minimizers of the Dirichlet integral)	
4.5	Prescribing Homological Singularities	543
	(s -degree. Lower bounds for the dipole energy)	
5	Notes	546

6.	The Non Parametric Area Functional	563
1	Area Minimizing Hypersurfaces	564
1.1	Parametric Surfaces of Least Area (Hypersurfaces as Caccioppoli's boundaries: De Giorgi's regularity theorem, monotonicity, Federer's regularity theorem. Surfaces as rectifiable currents: Almgren's regularity theorem. Minimal surfaces as stationary varifolds: a survey of Allard theory. Boundary regularity: Allard's and Hardt Simon's results)	564
1.2	Non Parametric Minimal Surfaces of Codimension One (Solvability of the Dirichlet problem. Bombieri-De Giorgi-Miranda a priori estimate. The variational approach. Removable singularities. Liouville type theorems. Bernstein theorem. Bombieri-De Giorgi-Giusti theorem on minimal cones)	579
2	Problems for Maps of Bounded Variation with Values in S^1	590
2.1	Preliminaries (Forms and currents in $\Omega \times S^1$. $BV(\Omega, \mathbb{R})$: a survey of results)	594
2.2	The Class $\text{cart}(\Omega \times S^1)$ (The structure and the approximation theorems)	600
2.3	Relaxed Energies and Existence of Minimizers (The area integral for maps into S^1 : the relaxed area. Minimizers in $\text{cart}(\Omega \times S^1)$. Dipole-type problems)	610
3	Two Dimensional Minimal Surfaces	619
3.1	Plateau's Problem (Morrey's ε -conformality theorem. Douglas-Rado existence theorem. Hildebrandt's boundary regularity theorem. Branch points and embedded minimal surfaces: Fleming, Meeks-Yau, and Chang results)	619
3.2	Existence of Two Dimensional Non Parametric Minimal Surfaces (Radò's theorem and existence, uniqueness, and regularity of two-dimensional graphs of any codimension)	625
3.3	The Minimal Surface System (Stationary graphs are not necessarily area minimizing. Existence and non existence of stationary Lipschitz graphs. Isolated singularities are not removable in high codimension. A Bernstein type result of Hildebrandt, Jost, and Widman)	627
4	Least Area Mappings and Least Mass Currents	632
4.1	Topological Results (Representation and homology of Lipschitz chains)	633
4.2	Main Results (Least area mapping $u : B^n \rightarrow \mathbb{R}^n$ and least mass currents "agree" if $n \geq 3$. If $n \geq 3$ the homotopy least area problem reduces to the homology problem)	635
5	The Non-parametric Area Integral	639
5.1	The Mass of Cartesian Currents and the Relaxed Area (Graphs of finite mass which cannot be approximated in area by smooth graphs)	641
5.2	Lebesgue's Area (The mass of 2-dimensional continuous Cartesian maps is Lebesgue's area of their graphs)	649
6	Notes	651
	Bibliography	653
	Index	683
	Symbols	695