CONTENTS

Chapter 1
INTRODUCTION

1.1. How Do Extremal Problems Appear?. 1
1.1.1. The Classical Isoperimetric Problem. Dido's Problem. 1
1.1.2. Some Other Ancient Extremal Problems in Geometry . . 5
1.1.3. Fermat's Variational Principle and Huygens' Principle. The Light Refraction Problem 7
1.1.4. The Brachistochrone Problem. The Origin of the Calculus of Variations 9
1.1.5. Newton's Aerodynamic Problem 11
1.1.6. The Diet Problem and the Transportation Problem. . . 12
1.1.7. The Time-Optimal Problem 13

1.2. How Are Extremal Problems Formalized? 13
1.2.1. Basic Definitions. 13
1.2.2. The Simplest Examples of Formalization of Extremal Problems 14
1.2.3. The Formalization of Newton's Problem. 15
1.2.4. Various Formalizations of the Classical Isoperimetric Problem and the Brachistochrone Problem. The Simplest Time-Optimal Problem. 17
1.2.5. Formalization of the Transportation and the Diet Problems 19
1.2.6. The Basic Classes of Extremal Problems 19

1.3. Lagrange Multiplier Rule and the Kuhn–Tucker Theorem. . . . 23
1.3.1. The Theorem of Fermat. 23
1.3.2. The Lagrange Multiplier Rule 24
1.3.3. The Kuhn–Tucker Theorem. 28
1.3.4. Proof of the Finite-Dimensional Separation Theorem. 31

1.4. Simplest Problem of the Classical Calculus of Variations and Its Generalizations 32
1.4.1. Euler Equation 32
1.4.2. Necessary Conditions in the Bolza Problem. Transversality Conditions. 36
1.4.3. The Extension of the Simplest Problem. 37

1.4.4. Needlelike Variations. Weierstrass' Condition . . . 43
1.4.5. Isoperimetric Problem and the Problem with Higher Derivatives 45

1.5. Lagrange Problem and the Basic Optimal Control Problem. . . 47
1.5.1. Statements of the Problems 47
1.5.2. Necessary Conditions in the Lagrange Problem 48
1.5.3. The Pontryagin Maximum Principle 50
1.5.4. Proof of the Maximum Principle for the Problem with a Free End Point. 51

1.6. Solutions of the Problems 56
1.6.1. Geometrical Extremal Problems. 57
1.6.2. Newton's Aerodynamic Problem 60
1.6.3. The Simplest Time-Optimal Problem. 63
1.6.4. The Classical Isoperimetric Problem and the Chaplygin Problem. 66
1.6.5. The Brachistochrone Problem and Some Geometrical Problems 69

Chapter 2
MATHEMATICAL METHODS OF THE THEORY OF EXTREMAL PROBLEMS

2.1. Background Material of Functional Analysis. 71
2.1.1. Normed Linear Spaces and Banach Spaces 71
2.1.2. Product Space. Factor Space. 72
2.1.3. Hahn–Banach Theorem and Its Corollaries 74
2.1.4. Separation Theorems. 77
2.1.5. Inverse Operator Theorem of Banach and the Lemma on the Right Inverse Mapping 79
2.1.6. Lemma on the Closed Image. 80
2.1.7. Lemma on the Annihilator of the Kernel of a Regular Operator 81
2.1.8. Absolutely Continuous Functions. 81
2.1.9. Riesz' Representation Theorem for the General Linear Functional on the Space C. Dirichlet's Formula. 84

2.2. Fundamentals of Differential Calculus in Normed Linear Spaces. 85
2.2.1. Directional Derivative, First Variation, the Fréchet and the Gateaux Derivatives, Strict Differentiability. 85
2.2.2. Theorem on the Composition of Differentiable Mappings 90
2.2.3. Mean Value Theorem and Its Corollaries 92
2.2.4. Differentiation in a Product Space. Partial Derivatives. The Theorem on the Total Differential 94
2.2.5. Higher-Order Derivatives. Taylor's Formula. 97

2.3. Implicit Function Theorem 101
2.3.1. Statement of the Existence Theorem for an Implicit Function. 101
2.3.2. Modified Contraction Mapping Principle 102
2.3.3. Proof of the Theorem 103
2.3.4. The Classical Implicit Function and Inverse Mapping Theorems 105

2.3.5. Tangent Space and Lyusternik's Theorem 108

2.4. Differentiability of Certain Concrete Mappings. 111
2.4.1. Nemytskii's Operator and a Differential Constraint Operator 111
2.4.2. Integral Functional. 113
2.4.3. Operator of Boundary Conditions. 115

2.5. Necessary Facts of the Theory of Ordinary Differential Equations 116
2.5.1. Basic Assumptions. 117
2.5.2. Local Existence Theorem. 118
2.5.3. Uniqueness Theorem 120
2.5.4. Linear Differential Equations. 121
2.5.5. Global Existence and Continuity Theorem. 124
2.5.6. Theorem on the Differentiability of Solutions with Respect to the Initial Data 128
2.5.7. Classical Theorem on the Differentiability of Solutions with Respect to the Initial Data 130

2.6. Elements of Convex Analysis 133
2.6.1. Basic Definitions. 133
2.6.2. Convex Sets and Functions in Topological Linear Spaces 138
2.6.3. Legendre–Young–Fenchel Transform. The Fenchel–Moreau Theorem 144
2.6.4. Subdifferential. Moreau–Rockafellar Theorem. Dubovitskii–Milyutin Theorem 147

Chapter 3
THE LAGRANGE PRINCIPLE FOR SMOOTH PROBLEMS WITH CONSTRAINTS

3.1. Elementary Problems 154
3.1.1. Elementary Problem without Constraints 154
3.1.2. Elementary Linear Programming Problem. 157
3.1.3. The Bolza Problem. 158
3.1.4. Elementary Optimal Control Problem 160
3.1.5. Lagrange Principle for Problems with Equality and Inequality Constraints 160

3.2. Lagrange Principle for Smooth Problems with Equality and Inequality Constraints. 163
3.2.1. Statement of the Theorem 163
3.2.2. Lagrange Multiplier Rule for Smooth Problems with Equality Constraints 164
3.2.3. Reduction of the Problem 165
3.2.4. Proof of the Theorem 166

3.3. Lagrange Principle and the Duality in Convex Programming Problems. 169
3.3.1. Kuhn–Tucker Theorem (Subdifferential Form) 169
3.3.2. Perturbation Method and the Duality Theorem. 170
3.3.3. Linear Programming: The Existence Theorem and the Duality Theorem. 174
3.3.4. Duality Theorem for the Shortest Distance Problem. Hoffman's Lemma and Minimax Lemma. 178

3.4. Second-Order Necessary Conditions and Sufficient Conditions for Extremum in Smooth Problems 186
3.4.1. Smooth Problems with Equality Constraints. 186
3.4.2. Second-Order Necessary Conditions for Smooth Problems with Equality and Inequality Constraints. . . . 188
3.4.3. Sufficient Conditions for an Extremum for Smooth Problems with Equality and Inequality Constraints. . 190

3.5. Application of the Theory to Algebra and Mathematical Analysis. 193
3.5.1. Fundamental Theorem of Algebra 193
3.5.2. Sylvester's Theorem. 194
3.5.3. Distance from a Point to a Subspace. Theorem on the Orthogonal Complement. Gram Determinants . . 195
3.5.4. Reduction of a Quadratic Form to Its Principal Axes 197
3.5.5. Legendre Quadratic Forms 201

Chapter 4
THE LAGRANGE PRINCIPLE FOR PROBLEMS OF THE CLASSICAL CALCULUS OF VARIATIONS AND OPTIMAL CONTROL THEORY

4.1. Lagrange Principle for the Lagrange Problem 203
4.1.1. Statement of the Problem and the Formulation of the Theorem. 203
4.1.2. Reduction of the Lagrange Problem to a Smooth Problem. 207
4.1.3. Generalized DuBois–Reymond Lemma 208
4.1.4. Derivation of Stationarity Conditions. 210
4.1.5. Problem with Higher-Order Derivatives. The Euler–Poisson Equation 212

4.2. The Pontryagin Maximum Principle. 214
4.2.1. Statement of the Optimal Control Problem 214
4.2.2. Formulation of the Maximum Principle. The Lagrange Principle for the Optimal Control Problem . 217
4.2.3. Needlelike Variations. 220
4.2.4. Reduction to the Finite-Dimensional Problem. 222
4.2.5. Proof of the Maximum Principle 223
4.2.6. Proof of the Lemma on the Packet of Needles. 227
4.2.7. Proof of the Lemma on the Integral Functionals . . . 234

4.3. Optimal Control Problems Linear with Respect to Phase Coordinates 236
4.3.1. Reduction of the Optimal Control Problem Linear with Respect to the Phase Coordinates to the Lyapunov-Type Problem. 236
4.3.2. Lyapunov's Theorem 237
4.3.3. Lagrange Principle for the Lyapunov-Type Problems 240
4.3.4. Duality Theorem. 245
4.3.5. Maximum Principle for Optimal Control Problems Linear with Respect to the Phase Coordinates 248

4.4. Application of the General Theory to the Simplest Problem of the Classical Calculus of Variations 250
4.4.1. Euler Equation. Weierstrass' Condition. Legendre's Condition 250

4.4.2. Second-Order Conditions for a Weak Extremum. Legendre's and Jacobi's Conditions 252
4.4.3. Hamiltonian Formalism. The Theorem on the Integral Invariant 255
4.4.4. Sufficient Conditions for an Absolute Extremum in the Simplest Problem 261
4.4.5. Conjugate Points. Sufficient Conditions for Strong and Weak Extrema 265
4.4.6. Theorem of A. E. Noether 272
4.4.7. Lagrange Variational Principle and the Conservation Laws in Mechanics 275

PROBLEMS 278
NOTES AND GUIDE TO THE LITERATURE. 298
REFERENCES 301
BASIC NOTATION 306