

# Contents

Prologue .....	1
Chapter 1. Introduction .....	4
1.1. Littlewood-Paley Theory for $\mathbb{T}$ .....	4
1.2. The LP and WM Properties .....	6
1.3. Extension of the LP and R Properties to Product Groups ...	20
1.4. Intersections of Decompositions Having the LP Property ...	28
Chapter 2. Convolution Operators (Scalar-Valued Case) .....	30
2.1. Covering Families .....	30
2.2. The Covering Lemma .....	32
2.3. The Decomposition Theorem .....	35
2.4. Bounds for Convolution Operators .....	41
Chapter 3. Convolution Operators (Vector-Valued Case) .....	50
3.1. Introduction .....	50
3.2. Vector-Valued Functions .....	50
3.3. Operator-Valued Kernels .....	52
3.4. Fourier Transforms .....	53
3.5. Convolution Operators .....	54
3.6. Bounds for Convolution Operators .....	55
Chapter 4. The Littlewood-Paley Theorem for Certain Disconnected Groups .....	57
4.1. The Littlewood-Paley Theorem for a Class of Totally Disconnected Groups .....	59
4.2. The Littlewood-Paley Theorem for a More General Class of Disconnected Groups .....	69
4.3. A Littlewood-Paley Theorem for Decompositions of $\mathbb{Z}$ Determined by a Decreasing Sequence of Subgroups .....	73
Chapter 5. Martingales and the Littlewood-Paley Theorem ...	76
5.1. Conditional Expectations .....	76
5.2. Martingales and Martingale Difference Series .....	80
5.3. The Littlewood-Paley Theorem .....	91
5.4. Applications to Disconnected Groups .....	100

Chapter 6. The Theorems of M. Riesz and Stečkin for $\mathbb{R}$ , $\mathbb{T}$ and $\mathbb{Z}$ .....	104
6.1. Introduction .....	104
6.2. The M. Riesz, Conjugate Function, and Stečkin Theorems for $\mathbb{R}$ .....	106
6.3. The M. Riesz, Conjugate Function, and Stečkin Theorems for $\mathbb{T}$ .....	111
6.4. The M. Riesz, Conjugate Function, and Stečkin Theorems for $\mathbb{Z}$ .....	114
6.5. The Vector Version of the M. Riesz Theorem for $\mathbb{R}$ , $\mathbb{T}$ and $\mathbb{Z}$ .....	118
6.6. The M. Riesz Theorem for $\mathbb{R}^k \times \mathbb{T}^m \times \mathbb{Z}^n$ .....	120
6.7. The Hilbert Transform .....	120
6.8. A Characterisation of the Hilbert Transform .....	128
Chapter 7. The Littlewood-Paley Theorem for $\mathbb{R}$ , $\mathbb{T}$ and $\mathbb{Z}$ : Dyadic Intervals .....	134
7.1. Introduction .....	134
7.2. The Littlewood-Paley Theorem: First Approach .....	136
7.3. The Littlewood-Paley Theorem: Second Approach .....	143
7.4. The Littlewood-Paley Theorem for Finite Products of $\mathbb{R}$ , $\mathbb{T}$ and $\mathbb{Z}$ : Dyadic Intervals .....	145
7.5. Fournier's Example .....	146
Chapter 8. Strong Forms of the Marcinkiewicz Multiplier Theorem and Littlewood-Paley Theorem for $\mathbb{R}$ , $\mathbb{T}$ and $\mathbb{Z}$ ...	148
8.1. Introduction .....	148
8.2. The Strong Marcinkiewicz Multiplier Theorem for $\mathbb{T}$ ...	148
8.3. The Strong Marcinkiewicz Multiplier Theorem for $\mathbb{R}$ ...	155
8.4. The Strong Marcinkiewicz Multiplier Theorem for $\mathbb{Z}$ ...	159
8.5. Decompositions which are not Hadamard .....	161
Chapter 9. Applications of the Littlewood-Paley Theorem ...	166
9.1. Some General Results .....	166
9.2. Construction of $\Lambda(p)$ Sets in $\mathbb{Z}$ .....	168
9.3. Singular Multipliers .....	172
Appendix A. Special Cases of the Marcinkiewicz Interpolation Theorem .....	177
A.1. The Concepts of Weak Type and Strong Type .....	177
A.2. The Interpolation Theorems .....	179
A.3. Vector-Valued Functions .....	183
Appendix B. The Homomorphism Theorem for Multipliers ...	184
B.1. The Key Lemmas .....	184
B.2. The Homomorphism Theorem .....	187

Appendix C. Harmonic Analysis on $\mathbb{D}_2$ and Walsh Series on $[0, 1]$ .....	193
Appendix D. Bernstein's Inequality .....	197
D.1. Bernstein's Inequality for $\mathbb{R}$ .....	197
D.2. Bernstein's Inequality for $\mathbb{T}$ .....	199
D.3. Bernstein's Inequality for LCA Groups .....	200
Historical Notes .....	202
References .....	206
Terminology .....	208
Index of Notation .....	209
Index of Authors and Subjects .....	211