

Contents

Introduction	xxi
Notation and Conventions	xxv

Part I Classical and Parabolic Potential Theory

Chapter I

Introduction to the Mathematical Background of Classical Potential Theory	3
1. The Context of Green's Identity	3
2. Function Averages	4
3. Harmonic Functions	4
4. Maximum-Minimum Theorem for Harmonic Functions	5
5. The Fundamental Kernel for \mathbb{R}^N and Its Potentials	6
6. Gauss Integral Theorem	7
7. The Smoothness of Potentials; The Poisson Equation	8
8. Harmonic Measure and the Riesz Decomposition	11

Chapter II

Basic Properties of Harmonic, Subharmonic, and Superharmonic Functions	14
1. The Green Function of a Ball; The Poisson Integral	14
2. Harnack's Inequality	16
3. Convergence of Directed Sets of Harmonic Functions	17
4. Harmonic, Subharmonic, and Superharmonic Functions	18
5. Minimum Theorem for Superharmonic Functions	20
6. Application of the Operation τ_B	20
7. Characterization of Superharmonic Functions in Terms of Harmonic Functions	22
8. Differentiable Superharmonic Functions	23
9. Application of Jensen's Inequality	23
10. Superharmonic Functions on an Annulus	24
11. Examples	25
12. The Kelvin Transformation ($N \geq 2$)	26

13. Greenian Sets	27
14. The $L^1(\mu_{B^-})$ and $D(\mu_{B^-})$ Classes of Harmonic Functions on a Ball B ; The Riesz–Herglotz Theorem	27
15. The Fatou Boundary Limit Theorem	31
16. Minimal Harmonic Functions	33

Chapter III

Infima of Families of Superharmonic Functions	35
1. Least Superharmonic Majorant (LM) and Greatest Subharmonic Minorant (GM)	35
2. Generalization of Theorem 1	36
3. Fundamental Convergence Theorem (Preliminary Version)	37
4. The Reduction Operation	38
5. Reduction Properties	41
6. A Smallness Property of Reductions on Compact Sets	42
7. The Natural (Pointwise) Order Decomposition for Positive Superharmonic Functions	43

Chapter IV

Potentials on Special Open Sets	45
1. Special Open Sets, and Potentials on Them	45
2. Examples	47
3. A Fundamental Smallness Property of Potentials	48
4. Increasing Sequences of Potentials	49
5. Smoothing of a Potential	49
6. Uniqueness of the Measure Determining a Potential	50
7. Riesz Measure Associated with a Superharmonic Function	51
8. Riesz Decomposition Theorem	52
9. Counterpart for Superharmonic Functions on \mathbb{R}^2 of the Riesz Decomposition	53
10. An Approximation Theorem	55

Chapter V

Polar Sets and Their Applications	57
1. Definition	57
2. Superharmonic Functions Associated with a Polar Set	58
3. Countable Unions of Polar Sets	59
4. Properties of Polar Sets	59
5. Extension of a Superharmonic Function	60
6. Greenian Sets in \mathbb{R}^2 as the Complements of Nonpolar Sets	63
7. Superharmonic Function Minimum Theorem (Extension of Theorem II.5)	63
8. Evans–Vasilesco Theorem	64
9. Approximation of a Potential by Continuous Potentials	66
10. The Domination Principle	67
11. The Infinity Set of a Potential and the Riesz Measure	68

Chapter VI

The Fundamental Convergence Theorem and the Reduction

Operation	70
1. The Fundamental Convergence Theorem	70
2. Inner Polar versus Polar Sets	71
3. Properties of the Reduction Operation	74
4. Proofs of the Reduction Properties	77
5. Reductions and Capacities	84

Chapter VII

Green Functions	85
1. Definition of the Green Function G_D	85
2. Extremal Property of G_D	87
3. Boundedness Properties of G_D	88
4. Further Properties of G_D	90
5. The Potential $G_D\mu$ of a Measure μ	92
6. Increasing Sequences of Open Sets and the Corresponding Green Function Sequences	94
7. The Existence of G_D versus the Greenian Character of D	94
8. From Special to Greenian Sets	95
9. Approximation Lemma	95
10. The Function $G_D(\cdot, \zeta)_{D-\zeta_1}$ as a Minimal Harmonic Function	96

Chapter VIII

The Dirichlet Problem for Relative Harmonic Functions	98
1. Relative Harmonic, Superharmonic, and Subharmonic Functions	98
2. The PWB Method	99
3. Examples	104
4. Continuous Boundary Functions on the Euclidean Boundary ($h \equiv 1$)	106
5. h -Harmonic Measure Null Sets	108
6. Properties of PWB ^h Solutions	110
7. Proofs for Section 6	111
8. h -Harmonic Measure	114
9. h -Resolutive Boundaries	118
10. Relations between Reductions and Dirichlet Solutions	122
11. Generalization of the Operator τ_B^h and Application to GM ^h	123
12. Barriers	124
13. h -Barriers and Boundary Point h -Regularity	126
14. Barriers and Euclidean Boundary Point Regularity	127
15. The Geometrical Significance of Regularity (Euclidean Boundary, $h \equiv 1$)	128
16. Continuation of Section 13	130
17. h -Harmonic Measure μ_D^h as a Function of D	131
18. The Extension G_D^- of G_D and the Harmonic Average $\mu_D(\zeta, G_B^-(\eta, \cdot))$ When $D \subset B$	132
19. Modification of Section 18 for $D = \mathbb{R}^2$	136
20. Interpretation of ϕ_D as a Green Function with Pole ∞ ($N = 2$)	139
21. Variant of the Operator τ_B	140

Chapter IX

Lattices and Related Classes of Functions	141
1. Introduction	141
2. $LM_D^h u$ for an h -Subharmonic Function u	141
3. The Class $\mathbf{D}(\mu_D^h)$	142
4. The Class $\mathbf{L}^p(\mu_D^h)$ ($p \geq 1$)	144
5. The Lattices (\mathbf{S}^+, \leq) and (\mathbf{S}^+, \leq)	145
6. The Vector Lattice (\mathbf{S}, \leq)	146
7. The Vector Lattice \mathbf{S}_m	148
8. The Vector Lattice \mathbf{S}_p	149
9. The Vector Lattice \mathbf{S}_{qb}	150
10. The Vector Lattice \mathbf{S}_s	151
11. A Refinement of the Riesz Decomposition	152
12. Lattices of h -Harmonic Functions on a Ball	152

Chapter X

The Sweeping Operation	155
1. Sweeping Context and Terminology	155
2. Relation between Harmonic Measure and the Sweeping Kernel	157
3. Sweeping Symmetry Theorem	158
4. Kernel Property of δ_D^A	158
5. Swept Measures and Functions	160
6. Some Properties of δ_D^A	161
7. Poles of a Positive Harmonic Function	163
8. Relative Harmonic Measure on a Polar Set	164

Chapter XI

The Fine Topology	166
1. Definitions and Basic Properties	166
2. A Thinness Criterion	168
3. Conditions That $\xi \in A^f$	169
4. An Internal Limit Theorem	171
5. Extension of the Fine Topology to $\mathbb{R}^N \cup \{\infty\}$	175
6. The Fine Topology Derived Set of a Subset of \mathbb{R}^N	177
7. Application to the Fundamental Convergence Theorem and to Reductions	177
8. Fine Topology Limits and Euclidean Topology Limits	178
9. Fine Topology Limits and Euclidean Topology Limits (Continued)	179
10. Identification of A^f in Terms of a Special Function $u^\#$	180
11. Quasi-Lindelöf Property	180
12. Regularity in Terms of the Fine Topology	181
13. The Euclidean Boundary Set of Thinness of a Greenian Set	182
14. The Support of a Swept Measure	183
15. Characterization of $\ \mu\ ^A$	183
16. A Special Reduction	184
17. The Fine Interior of a Set of Constancy of a Superharmonic Function	184
18. The Support of a Swept Measure (Continuation of Section 14)	185
19. Superharmonic Functions on Fine-Open Sets	187
20. A Generalized Reduction	187

21. Limits of Superharmonic Functions at Irregular Boundary Points of Their Domains	190
22. The Limit Harmonic Measure $\int \mu_D$	191
23. Extension of the Domination Principle	194

Chapter XII

The Martin Boundary	195
1. Motivation	195
2. The Martin Functions	196
3. The Martin Space	197
4. Preliminary Representations of Positive Harmonic Functions and Their Reductions	199
5. Minimal Harmonic Functions and Their Poles	200
6. Extension of Lemma 4	201
7. The Set of Nonminimal Martin Boundary Points	202
8. Reductions on the Set of Minimal Martin Boundary Points	203
9. The Martin Representation	204
10. Resolutivity of the Martin Boundary	207
11. Minimal Thinness at a Martin Boundary Point	208
12. The Minimal-Fine Topology	210
13. First Martin Boundary Counterpart of Theorem XI.4(c) and (d)	213
14. Second Martin Boundary Counterpart of Theorem XI.4(c)	213
15. Minimal-Fine Topology Limits and Martin Topology Limits at a Minimal Martin Boundary Point	215
16. Minimal-Fine Topology Limits and Martin Topology Limits at a Minimal Martin Boundary Point (Continued)	216
17. Minimal-Fine Martin Boundary Limit Functions	216
18. The Fine Boundary Function of a Potential	218
19. The Fatou Boundary Limit Theorem for the Martin Space	219
20. Classical versus Minimal-Fine Topology Boundary Limit Theorems for Relative Superharmonic Functions on a Ball in \mathbb{R}^N	221
21. Nontangential and Minimal-Fine Limits at a Half-space Boundary	222
22. Normal Boundary Limits for a Half-space	223
23. Boundary Limit Function (Minimal-Fine and Normal) of a Potential on a Half-space	225

Chapter XIII

Classical Energy and Capacity	226
1. Physical Context	226
2. Measures and Their Energies	227
3. Charges and Their Energies	228
4. Inequalities between Potentials, and the Corresponding Energy Inequalities	229
5. The Function $D \mapsto G_D \mu$	230
6. Classical Evaluation of Energy; Hilbert Space Methods	231
7. The Energy Functional (Relative to an Arbitrary Greenian Subset D of \mathbb{R}^N).	233
8. Alternative Proofs of Theorem 7(b^+)	235

9. Sharpening of Lemma 4	237
10. The Classical Capacity Function	237
11. Inner and Outer Capacities (Notation of Section 10).....	240
12. Extremal Property Characterizations of Equilibrium Potentials (Notation of Section 10).....	241
13. Expressions for $C(A)$	243
14. The Gauss Minimum Problems and Their Relation to Reductions.....	244
15. Dependence of C^* on D	247
16. Energy Relative to \mathbb{R}^2	248
17. The Wiener Thinness Criterion.....	249
18. The Robin Constant and Equilibrium Measures Relative to \mathbb{R}^2 ($N = 2$) ..	251

Chapter XIV

One-Dimensional Potential Theory	256
1. Introduction.....	256
2. Harmonic, Superharmonic, and Subharmonic Functions.....	256
3. Convergence Theorems	256
4. Smoothness Properties of Superharmonic and Subharmonic Functions... ..	257
5. The Dirichlet Problem (Euclidean Boundary).....	257
6. Green Functions	258
7. Potentials of Measures	259
8. Identification of the Measure Defining a Potential	259
9. Riesz Decomposition	260
10. The Martin Boundary	261

Chapter XV

Parabolic Potential Theory: Basic Facts	262
1. Conventions.....	262
2. The Parabolic and Coparabolic Operators	263
3. Coparabolic Polynomials.....	264
4. The Parabolic Green Function of \mathbb{R}^N	266
5. Maximum-Minimum Parabolic Function Theorem.....	267
6. Application of Green's Theorem	269
7. The Parabolic Green Function of a Smooth Domain; The Riesz Decomposition and Parabolic Measure (Formal Treatment)	270
8. The Green Function of an Interval.....	272
9. Parabolic Measure for an Interval	273
10. Parabolic Averages	275
11. Harnack's Theorems in the Parabolic Context	276
12. Superparabolic Functions	277
13. Superparabolic Function Minimum Theorem	279
14. The Operation $\tau_{\bar{b}}$ and the Defining Average Properties of Superparabolic Functions	280
15. Superparabolic and Parabolic Functions on a Cylinder	281
16. The Appell Transformation	282
17. Extensions of a Parabolic Function Defined on a Cylinder	283

Chapter XVI

Subparabolic, Superparabolic, and Parabolic Functions on a Slab . . . 285

1. The Parabolic Poisson Integral for a Slab 285
2. A Generalized Superparabolic Function Inequality 287
3. A Criterion of a Subparabolic Function Supremum 288
4. A Boundary Limit Criterion for the Identically Vanishing of a Positive Parabolic Function 288
5. A Condition that a Positive Parabolic Function Be Representable by a Poisson Integral 290
6. The $L^1(\mu_{\tilde{B}})$ and $D(\mu_{\tilde{B}})$ Classes of Parabolic Functions on a Slab 290
7. The Parabolic Boundary Limit Theorem 292
8. Minimal Parabolic Functions on a Slab 293

Chapter XVII

Parabolic Potential Theory (Continued) 295

1. Greatest Minorants and Least Majorants 295
2. The Parabolic Fundamental Convergence Theorem (Preliminary Version) and the Reduction Operation 295
3. The Parabolic Context Reduction Operations 296
4. The Parabolic Green Function 298
5. Potentials 300
6. The Smoothness of Potentials 303
7. Riesz Decomposition Theorem 305
8. Parabolic-Polar Sets 305
9. The Parabolic-Fine Topology 308
10. Semipolar Sets 309
11. Preliminary List of Reduction Properties 310
12. A Criterion of Parabolic Thinness 313
13. The Parabolic Fundamental Convergence Theorem 314
14. Applications of the Fundamental Convergence Theorem to Reductions and to Green Functions 316
15. Applications of the Fundamental Convergence Theorem to the Parabolic-Fine Topology 317
16. Parabolic-Reduction Properties 317
17. Proofs of the Reduction Properties in Section 16 320
18. The Classical Context Green Function in Terms of the Parabolic Context Green Function ($N \geq 1$) 326
19. The Quasi-Lindelöf Property 328

Chapter XVIII

The Parabolic Dirichlet Problem, Sweeping, and Exceptional Sets . . . 329

1. Relativization of the Parabolic Context; The PWB Method in this Context 329
2. \hbar -Parabolic Measure 332
3. Parabolic Barriers 333
4. Relations between the Classical Dirichlet Problem and the Parabolic Context Dirichlet Problem 334
5. Classical Reductions in the Parabolic Context 335

6. Parabolic Regularity of Boundary Points	337
7. Parabolic Regularity in Terms of the Fine Topology	341
8. Sweeping in the Parabolic Context	341
9. The Extension $\dot{G}_{\dot{D}}$ of $\dot{G}_{\dot{B}}$ and the Parabolic Average $\mu_{\dot{D}}(\dot{\zeta}, \dot{G}_{\dot{B}}^{\dot{\zeta}}(\cdot, \eta))$ when $\dot{D} \subset \dot{B}$	343
10. Conditions that $\dot{\zeta} \in \dot{A}^{pf}$	345
11. Parabolic- and Coparabolic-Polar Sets	347
12. Parabolic- and Coparabolic-Semipolar Sets	348
13. The Support of a Swept Measure	350
14. An Internal Limit Theorem; The Coparabolic-Fine Topology Smoothness of Superparabolic Functions	351
15. Application to a Version of the Parabolic Context Fatou Boundary Limit Theorem on a Slab	357
16. The Parabolic Context Domination Principle	358
17. Limits of Superparabolic Functions at Parabolic-Irregular Boundary Points of Their Domains	358
18. Martin Flat Point Set Pairs	361
19. Lattices and Related Classes of Functions in the Parabolic Context	361

Chapter XIX

The Martin Boundary in the Parabolic Context	363
1. Introduction	363
2. The Martin Functions of Martin Point Set and Measure Set Pairs	364
3. The Martin Space \dot{D}^M	366
4. Preparatory Material for the Parabolic Context Martin Representation Theorem	367
5. Minimal Parabolic Functions and Their Poles	369
6. The Set of Nonminimal Martin Boundary Points	370
7. The Martin Representation in the Parabolic Context	371
8. Martin Boundary of a Slab $\dot{D} = \mathbb{R}^N \times]0, \delta[$ with $0 < \delta \leq +\infty$	371
9. Martin Boundaries for the Lower Half-space of \mathbb{R}^N and for \mathbb{R}^N	374
10. The Martin Boundary of $\dot{D} =]0, +\infty[\times]-\infty, \delta[$	375
11. \dot{PWB}^h Solutions on \dot{D}^M	377
12. The Minimal-Fine Topology in the Parabolic Context	377
13. Boundary Counterpart of Theorem XVIII.14(f)	379
14. The Vanishing of Potentials on $\partial^M \dot{D}$	381
15. The Parabolic Context Fatou Boundary Limit Theorem on Martin Spaces	381

Part 2

Probabilistic Counterpart of Part 1

Chapter I

Fundamental Concepts of Probability	387
1. Adapted Families of Functions on Measurable Spaces	387
2. Progressive Measurability	388
3. Random Variables	390

- 4. Conditional Expectations 391
- 5. Conditional Expectation Continuity Theorem 393
- 6. Fatou's Lemma for Conditional Expectations 396
- 7. Dominated Convergence Theorem for Conditional Expectations 397
- 8. Stochastic Processes, "Evanescant," "Indistinguishable," "Standard Modification," "Nearly" 398
- 9. The Hitting of Sets and Progressive Measurability 401
- 10. Canonical Processes and Finite-Dimensional Distributions 402
- 11. Choice of the Basic Probability Space 404
- 12. The Hitting of Sets by a Right Continuous Process 405
- 13. Measurability versus Progressive Measurability of Stochastic Processes 407
- 14. Predictable Families of Functions 410

Chapter II

- Optional Times and Associated Concepts 413
 - 1. The Context of Optional Times 413
 - 2. Optional Time Properties (Continuous Parameter Context). 415
 - 3. Process Functions at Optional Times 417
 - 4. Hitting and Entry Times 419
 - 5. Application to Continuity Properties of Sample Functions 421
 - 6. Continuation of Section 5 423
 - 7. Predictable Optional Times 423
 - 8. Section Theorems 425
 - 9. The Graph of a Predictable Time and the Entry Time of a Predictable Set 426
- 10. Semipolar Subsets of $\mathbb{R}^+ \times \Omega$ 427
- 11. The Classes **D** and L^p of Stochastic Processes 428
- 12. Decomposition of Optional Times; Accessible and Totally Inaccessible Optional Times 429

Chapter III

- Elements of Martingale Theory 432
 - 1. Definitions 432
 - 2. Examples 433
 - 3. Elementary Properties (Arbitrary Simply Ordered Parameter Set) 435
 - 4. The Parameter Set in Martingale Theory 437
 - 5. Convergence of Supermartingale Families 437
 - 6. Optional Sampling Theorem (Bounded Optional Times) 438
 - 7. Optional Sampling Theorem for Right Closed Processes 440
 - 8. Optional Stopping 442
 - 9. Maximal Inequalities 442
 - 10. Conditional Maximal Inequalities 444
 - 11. An L^p Inequality for Submartingale Suprema 444
 - 12. Crossings 445
 - 13. Forward Convergence in the L^1 Bounded Case 450
 - 14. Convergence of a Uniformly Integrable Martingale 451
 - 15. Forward Convergence of a Right Closable Supermartingale 453
 - 16. Backward Convergence of a Martingale 454

17. Backward Convergence of a Supermartingale	455
18. The τ Operator	455
19. The Natural Order Decomposition Theorem for Supermartingales	457
20. The Operators LM and GM	458
21. Supermartingale Potentials and the Riesz Decomposition	459
22. Potential Theory Reductions in a Discrete Parameter Probability Context	459
23. Application to the Crossing Inequalities	461

Chapter IV

Basic Properties of Continuous Parameter Supermartingales	463
1. Continuity Properties	463
2. Optional Sampling of Uniformly Integrable Continuous Parameter Martingales	468
3. Optional Sampling and Convergence of Continuous Parameter Supermartingales	470
4. Increasing Sequences of Supermartingales	473
5. Probability Version of the Fundamental Convergence Theorem of Potential Theory	476
6. Quasi-Bounded Positive Supermartingales; Generation of Supermartingale Potentials by Increasing Processes	480
7. Natural versus Predictable Increasing Processes ($I = \mathbb{Z}^+$ or \mathbb{R}^+)	483
8. Generation of Supermartingale Potentials by Increasing Processes in the Discrete Parameter Case	488
9. An Inequality for Predictable Increasing Processes	489
10. Generation of Supermartingale Potentials by Increasing Processes for Arbitrary Parameter Sets	490
11. Generation of Supermartingale Potentials by Increasing Processes in the Continuous Parameter Case: The Meyer Decomposition	493
12. Meyer Decomposition of a Submartingale	495
13. Role of the Measure Associated with a Supermartingale; The Supermartingale Domination Principle	496
14. The Operators τ , LM, and GM in the Continuous Parameter Context.	500
15. Potential Theory on $\mathbb{R}^+ \times \Omega$	501
16. The Fine Topology of $\mathbb{R}^+ \times \Omega$	502
17. Potential Theory Reductions in a Continuous Parameter Probability Context	504
18. Reduction Properties	505
19. Proofs of the Reduction Properties in Section 18	509
20. Evaluation of Reductions	513
21. The Energy of a Supermartingale Potential.	515
22. The Subtraction of a Supermartingale Discontinuity	516
23. Supermartingale Decompositions and Discontinuities	518

Chapter V

Lattices and Related Classes of Stochastic Processes	520
1. Conventions; The Essential Order	520
2. LM $x(\cdot)$ when $\{x(\cdot), \mathcal{F}(\cdot)\}$ Is a Submartingale	521

3. Uniformly Integrable Positive Submartingales	523
4. L^p Bounded Stochastic Processes ($p \geq 1$)	524
5. The Lattices (\mathbf{S}^\pm, \leq) , (\mathbf{S}^+, \leq) , (\mathbf{S}^\pm, \leq) , (\mathbf{S}^+, \leq)	525
6. The Vector Lattices (\mathbf{S}, \leq) and (\mathbf{S}, \leq)	528
7. The Vector Lattices (\mathbf{S}_m, \leq) and (\mathbf{S}_m, \leq)	529
8. The Vector Lattices (\mathbf{S}_p, \leq) and (\mathbf{S}_p, \leq)	530
9. The Vector Lattices (\mathbf{S}_{qb}, \leq) and (\mathbf{S}_{qb}, \leq)	531
10. The Vector Lattices (\mathbf{S}_s, \leq) and (\mathbf{S}_s, \leq)	532
11. The Orthogonal Decompositions $\mathbf{S}_m = \mathbf{S}_{mqb} + \mathbf{S}_{ms}$ and $\mathbf{S}_m = \mathbf{S}_{mqb} + \mathbf{S}_{ms}$	533
12. Local Martingales and Singular Supermartingale Potentials in (\mathbf{S}, \leq)	534
13. Quasimartingales (Continuous Parameter Context)	535

Chapter VI

Markov Processes	539
1. The Markov Property	539
2. Choice of Filtration	544
3. Integral Parameter Markov Processes with Stationary Transition Probabilities	545
4. Application of Martingale Theory to Discrete Parameter Markov Processes	547
5. Continuous Parameter Markov Processes with Stationary Transition Probabilities	550
6. Specialization to Right Continuous Processes	552
7. Continuous Parameter Markov Processes: Lifetimes and Trap Points	554
8. Right Continuity of Markov Process Filtrations; A Zero-One (0-1) Law	556
9. Strong Markov Property	557
10. Probabilistic Potential Theory; Excessive Functions	560
11. Excessive Functions and Supermartingales	564
12. Excessive Functions and the Hitting Times of Analytic Sets (Notation and Hypotheses of Section 11)	565
13. Conditioned Markov Processes	566
14. Tied Down Markov Processes	567
15. Killed Markov Processes	568

Chapter VII

Brownian Motion	570
1. Processes with Independent Increments and State Space \mathbb{R}^N	570
2. Brownian Motion	572
3. Continuity of Brownian Paths	576
4. Brownian Motion Filtrations	578
5. Elementary Properties of the Brownian Transition Density and Brownian Motion	581
6. The Zero-One Law for Brownian Motion	583
7. Tied Down Brownian Motion	586
8. André Reflection Principle	587
9. Brownian Motion in an Open Set ($N \geq 1$)	589
10. Space-Time Brownian Motion in an Open Set	592
11. Brownian Motion in an Interval	594

12. Probabilistic Evaluation of Parabolic Measure for an Interval	595
13. Probabilistic Significance of the Heat Equation and Its Dual	596
 Chapter VIII	
The Itô Integral	599
1. Notation	599
2. The Size of Γ_0	601
3. Properties of the Itô Integral	602
4. The Stochastic Integral for an Integrand Process in Γ_0	605
5. The Stochastic Integral for an Integrand Process in Γ	606
6. Proofs of the Properties in Section 3	607
7. Extension to Vector-Valued and Complex-Valued Integrands	611
8. Martingales Relative to Brownian Motion Filtrations	612
9. A Change of Variables	615
10. The Role of Brownian Motion Increments	618
11. ($N = 1$) Computation of the Itô Integral by Riemann–Stieltjes Sums	620
12. Itô's Lemma	621
13. The Composition of the Basic Functions of Potential Theory with Brownian Motion	625
14. The Composition of an Analytic Function with Brownian Motion	626
 Chapter IX	
Brownian Motion and Martingale Theory	627
1. Elementary Martingale Applications	627
2. Coparabolic Polynomials and Martingale Theory	630
3. Superharmonic and Harmonic Functions on \mathbb{R}^N and Supermartingales and Martingales	632
4. Hitting of an F_σ Set	635
5. The Hitting of a Set by Brownian Motion	636
6. Superharmonic Functions, Excessive for Brownian Motion	637
7. Preliminary Treatment of the Composition of a Superharmonic Function with Brownian Motion; A Probabilistic Fatou Boundary Limit Theorem	641
8. Excessive and Invariant Functions for Brownian Motion	645
9. Application to Hitting Probabilities and to Parabolicity of Transition Densities	647
10. ($N = 2$). The Hitting of Nonpolar Sets by Brownian Motion	648
11. Continuity of the Composition of a Function with Brownian Motion	649
12. Continuity of Superharmonic Functions on Brownian Motion	650
13. Preliminary Probabilistic Solution of the Classical Dirichlet Problem	651
14. Probabilistic Evaluation of Reductions	653
15. Probabilistic Description of the Fine Topology	656
16. α -Excessive Functions for Brownian Motion and Their Composition with Brownian Motions	659
17. Brownian Motion Transition Functions as Green Functions; The Corresponding Backward and Forward Parabolic Equations	661
18. Excessive Measures for Brownian Motion	663
19. Nearly Borel Sets for Brownian Motion	666
20. Brownian Motion into a Set from an Irregular Boundary Point	666

Chapter X

Conditional Brownian Motion	668
1. Definition	668
2. h -Brownian Motion in Terms of Brownian Motion	671
3. Contexts for (2.1)	676
4. Asymptotic Character of h -Brownian Paths at Their Lifetimes	677
5. h -Brownian Motion from an Infinity of h	680
6. Brownian Motion under Time Reversal	682
7. Preliminary Probabilistic Solution of the Dirichlet Problem for h -Harmonic Functions; h -Brownian Motion Hitting Probabilities and the Corresponding Generalized Reductions	684
8. Probabilistic Boundary Limit and Internal Limit Theorems for Ratios of Strictly Positive Superharmonic Functions	688
9. Conditional Brownian Motion in a Ball	691
10. Conditional Brownian Motion Last Hitting Distributions; The Capacity Distribution of a Set in Terms of a Last Hitting Distribution	693
11. The Tail σ Algebra of a Conditional Brownian Motion	694
12. Conditional Space-Time Brownian Motion	699
13. [Space-Time] Brownian Motion in $[\mathbb{R}^N]$ \mathbb{R}^N with Parameter Set \mathbb{R}	700

Part 3

Chapter I

Lattices in Classical Potential Theory and Martingale Theory	705
1. Correspondence between Classical Potential Theory and Martingale Theory	705
2. Relations between Decomposition Components of \mathbf{S} in Potential Theory and Martingale Theory	706
3. The Classes \mathbf{L}^p and \mathbf{D}	706
4. PWB-Related Conditions on h -Harmonic Functions and on Martingales	707
5. Class \mathbf{D} Property versus Quasi-Boundedness	708
6. A Condition for Quasi-Boundedness	709
7. Singularity of an Element of \mathbf{S}_m^+	710
8. The Singular Component of an Element of \mathbf{S}^+	711
9. The Class \mathbf{S}_{pqb}	712
10. The Class \mathbf{S}_{ps}	714
11. Lattice Theoretic Analysis of the Composition of an h -Superharmonic Function with an h -Brownian Motion	715
12. A Decomposition of \mathbf{S}_{ms}^+ (Potential Theory Context)	716
13. Continuation of Section 11	717

Chapter II

Brownian Motion and the PWB Method	719
1. Context of the Problem	719
2. Probabilistic Analysis of the PWB Method	720

3. PWB ^h Examples	723
4. Tail σ Algebras in the PWB ^h Context	725
 Chapter III	
Brownian Motion on the Martin Space	727
1. The Structure of Brownian Motion on the Martin Space	727
2. Brownian Motions from Martin Boundary Points (Notation of Section 1)	728
3. The Zero-One Law at a Minimal Martin Boundary Point and the Probabilistic Formulation of the Minimal-Fine Topology (Notation of Section 1)	730
4. The Probabilistic Fatou Theorem on the Martin Space	732
5. Probabilistic Approach to Theorem I.XI.4(c) and Its Boundary Counterparts	733
6. Martin Representation of Harmonic Functions in the Parabolic Context	735
 Appendixes	
 Appendix I	
Analytic Sets	741
1. Pavings and Algebras of Sets	741
2. Suslin Schemes	741
3. Sets Analytic over a Product Paving	742
4. Analytic Extensions versus σ Algebra Extensions of Pavings	743
5. Projection Characterization $\mathcal{A}(\mathcal{Y})$	743
6. The Operation $\mathcal{A}(\mathcal{A})$	744
7. Projections of Sets in Product Pavings	744
8. Extension of a Measurability Concept to the Analytic Operation Context	745
9. The G_δ Sets of a Complete Metric Space	745
10. Polish Spaces	746
11. The Baire Null Space	746
12. Analytic Sets	747
13. Analytic Subsets of Polish Spaces	748
 Appendix II	
Capacity Theory	750
1. Choquet Capacities	750
2. Sierpinski Lemma	750
3. Choquet Capacity Theorem	751
4. Lusin's Theorem	751
5. A Fundamental Example of a Choquet Capacity	752
6. Strongly Subadditive Set Functions	752
7. Generation of a Choquet Capacity by a Positive Strongly Subadditive Set Function	753
8. Topological Precapacities	755
9. Universally Measurable Sets	756

Appendix III

Lattice Theory 758

1. Introduction 758
2. Lattice Definitions 758
3. Cones 758
4. The Specific Order Generated by a Cone 759
5. Vector Lattices 760
6. Decomposition Property of a Vector Lattice 762
7. Orthogonality in a Vector Lattice 762
8. Bands in a Vector Lattice 762
9. Projections on Bands 763
10. The Orthogonal Complement of a Set 764
11. The Band Generated by a Single Element 764
12. Order Convergence 765
13. Order Convergence on a Linearly Ordered Set 766

Appendix IV

Lattice Theoretic Concepts in Measure Theory 767

1. Lattices of Set Algebras 767
2. Measurable Spaces and Measurable Functions 767
3. Composition of Functions 768
4. The Measure Lattice of a Measurable Space 769
5. The σ Finite Measure Lattice of a Measurable Space (Notation of Section 4) 771
6. The Hahn and Jordan Decompositions 772
7. The Vector Lattice \mathcal{M}_σ 772
8. Absolute Continuity and Singularity 773
9. Lattices of Measurable Functions on a Measure Space 774
10. Order Convergence of Families of Measurable Functions 775
11. Measures on Polish Spaces 777
12. Derivates of Measures 778

Appendix V

Uniform Integrability 779

Appendix VI

Kernels and Transition Functions 781

1. Kernels 781
2. Universally Measurable Extension of a Kernel 782
3. Transition Functions 782

Appendix VII

Integral Limit Theorems 785

1. An Elementary Limit Theorem 785
2. Ratio Integral Limit Theorems 786
3. A One-Dimensional Ratio Integral Limit Theorem 786
4. A Ratio Integral Limit Theorem Involving Convex Variational Derivates 788

Appendix VIII	
Lower Semicontinuous Functions	791
1. The Lower Semicontinuous Smoothing of a Function	791
2. Suprema of Families of Lower Semicontinuous Functions	791
3. Choquet Topological Lemma	792
Historical Notes	793
Part 1	793
Part 2	806
Part 3	815
Appendixes	816
Bibliography	819
Notation Index	827
Index	829