

TABLE OF CONTENTS

Preface	v
Preface to the Second Edition	vi
Introduction	1

PART I

ORDINARY DIFFERENCE EQUATIONS

Chapter 1. Difference Equations of First and Second Order.	
Examples of Difference Schemes	5
§ 1. Simplest difference equations	5
1. Difference equations (1). 2. Order of difference equations (8). 3. General solution of difference equations (8).	
Problems	11
§ 2. Difference equation of first order	12
1. Fundamental solution (12). 2. Conditions governing the boundedness of the fundamental solution (13). 3. Particular solution (14).	
Problems	16
§ 3. Difference equation of second order	17
1. General solution of the homogeneous equation (17).	
2. General solution of the inhomogeneous equation. Fundamental solution (21). 3. Estimate of the fundamental solution in terms of the coefficients of the difference equation (26).	
Problems	28

Chapter 2. Boundary-Value Problems for Equations of Second Order	31
§ 4. Formulation of problem. Good-conditioning criteria	31
1. Formulation of problem (31). 2. Definition of a well-conditioned problem (32). 3. Sufficient condition for a well-conditioned problem (34). 4. Criterion for a well-conditioned boundary-value problem with constant coefficients (35). 5. Criterion for a well-conditioned problem with variable coefficients (35). 6. Justification of the criterion for a well-conditioned boundary-value problem with constant coefficients (37). 7. General boundary-value problem for a system of difference equations (42).	
Problems	46
§ 5. Algorithm for the solution of boundary-value problems - forward elimination, back substitution (FEBS)	47
1. Description of forward elimination, back substitution (FEBS) (47). 2. Example of a computationally unstable algorithm (50).	
Problems	51
Chapter 3. Basis of the FEBS Method	53
§ 6. Properties of well-conditioned boundary-value problems	53
1. Bound for the solution of a boundary-value problem with perturbed coefficients (53). 2. Proof of the criterion for good-conditioning (57). 3. Properties of a well-conditioned problem (62).	
§ 7. Basis for the FEBS method in well-conditioned boundary-value problems.....	63
1. Bounds on the FEBS coefficients (63). 2. Estimate of the influence on computational results of rounding errors committed in the course of calculation (65).	

PART 2

DIFFERENCE SCHEMES FOR ORDINARY DIFFERENTIAL EQUATIONS

Chapter 4. Elementary Examples of Difference Schemes	71
§ 8. The concept of order of accuracy and approximation	71
1. Order of accuracy of a difference scheme (71).	
2. Speed of convergence of the solution of the difference equation (75).	
3. Order of approximation (77).	
§ 9. Unstable difference schemes	78
1. Techniques for approximating the derivative (78).	
2. Example of an unstable difference scheme (79).	
Chapter 5. Convergence of the Solutions of Difference Equations as a Consequence of Approximation and Stability	83
§ 10. Convergence of a difference scheme	83
1. Concept of a net and a net function (83).	
2. Convergent difference schemes (88).	
3. Proof of convergence of a difference scheme (91).	
Problems	93
§ 11. Approximation of a differential boundary-value problem by a difference scheme	94
1. The residual $\delta f^{(h)}$ (94).	
2. Computation of the residual (96).	
3. Approximation of order h^k (98).	
4. Examples (99).	
5. Splitting of difference schemes into subsystems (102).	
6. Replacement of derivatives by difference expressions (105).	
7. Other methods for constructing difference schemes (108).	
Problems	109
§ 12. Definition of the stability of a difference scheme. Convergence as a consequence of approximation and stability	109
1. Definition of stability (109).	
2. Connection between approximation, stability and convergence (112).	
3. Convergent difference scheme for an integral equation (118).	
§ 13. On the choice of a norm	120

§ 14. Sufficient condition for stability of difference schemes for the solution of the Cauchy problem	128
1. Introductory example (129). 2. Canonical form of a difference scheme (130). 3. Stability viewed as the boundedness of the norms of powers of the transition operator (132). 4. Examples of investigations of stability (134). 5. Non-uniqueness of the canonical form (141).	
Problems	143
§ 15. Necessary spectral criterion for stability	144
1. Boundedness of the norms of powers of the transition operator necessary for stability (145). 2. Spectral criterion for stability (146). 3. Discussion of the spectral stability criterion (147).	
Problems	153
§ 16. Roundoff errors	154
1. Errors in the coefficients (154). 2. Computational errors (157).	
§ 17. Quantitative aspects of stability	159
§ 18. Method for studying stability of nonlinear problems	166
Chapter 6. Widely-Used Difference Schemes	169
§ 19. Runga-Kutta and Adams schemes	169
1. Runga-Kutta scheme (170). 2. Adams schemes (172). 3. Note on stability (176). 4. Generalization to systems of equations (177).	
§ 20. Methods of solution for boundary-value problems	179
1. The shooting method (180). 2. The FEBS method (182). 3. Newton's method (183).	

PART 3

DIFFERENCE SCHEMES FOR PARTIAL DIFFERENTIAL EQUATIONS. BASIC CONCEPTS

Chapter 7. Simplest Examples of the Construction and Study of Difference Schemes	185
§ 21. Review and illustrations of basic definitions	185
1. Definition of convergence (185). 2. Definition of approximation (186). 3. Definition of stability (190). Problems	197
§ 22. Simplest methods for the construction of approximating difference schemes	198
1. Replacement of derivatives by difference relations (198). 2. Method of undetermined coefficients (206). 3. Schemes with recomputation, or "predictor-corrector" schemes (217). 4. On other examples (219). Problems	219
§ 23. Examples of the formulation of boundary conditions in the construction of difference schemes	221
Problems	226
§ 24. The Courant-Friedrichs-Levy condition, necessary for convergence	228
1. The Courant-Friedrichs-Levy condition (228). 2. Examples of difference schemes for the Cauchy problem (229). 3. Examples of difference schemes for the Dirichlet problem (235). Problems	238
Chapter 8. Some basic methods for the study of stability	241
§ 25. Spectral analysis of the Cauchy difference problem	241
1. Stability with respect to starting values (241). 2. Necessary spectral condition for stability (242). 3. Examples (244). 4. Integral representation of the solution (252). 5. Smoothing of the difference solution as a result of approximations viscosity (257). Problems	260

§ 26. Principle of frozen coefficients	261
1. Frozen coefficients at interior points (262).	
2. Criterion of Babenko and Gelfand (264).	
Problems	270
§ 27. Representation of the solution of some model problems in the form of finite Fourier series	272
1. Fourier series for net functions (272). 2. Represent- ation of the solutions of difference schemes for the heat equation on an interval (276). 3. Representation of the solution of difference schemes for the two- dimensional heat-conduction problem (279). 4. Represent- ation of the solution of a difference scheme for the vibrating string problem (282).	
Problems	285
§ 28. The maximum principle	286
1. Explicit difference scheme (286). 2. Implicit difference scheme (289). 3. Comparison of explicit and implicit difference schemes (291).	
Chapter 9. Difference Scheme Concepts in the Computation of Generalized Solutions	293
§ 29. The generalized solution	293
1. Mechanism generating discontinuities (294).	
2. Definition of the generalized solution (295).	
3. Condition on a line of discontinuity of a solution (296). 4. Decay of an arbitrary discontinuity (298).	
5. Other definitions of the generalized solution (299).	
§ 30. The construction of difference schemes	300
1. Schemes with artificial viscosity (301). 2. Method of characteristics (301). 3. Divergence difference schemes (303).	

PART 4

PROBLEMS WITH TWO SPACE VARIABLE

Chapter 10. The Concept of Difference Schemes with Splitting	309
§ 31. Construction of splitting schemes	309
Problems	314
§ 32. Economical difference schemes	314
Problems	323
§ 33. Splitting by physical factors	323
Chapter 11. Elliptic problems	325
§ 34. Simplest difference scheme for the Dirichlet problem	325
1. Approximation (326). 2. Stability (327).	
Problems	331
§ 35. Method of time-development	332
1. Idea of the method of time-development (332).	
2. Analysis of the explicit time-development scheme (335).	
3. The alternating-direction scheme (338). 4. Choice	
of accuracy (340). 5. Limits of applicability of	
methods (340).	
Problems	341
§ 36. Iteration with variable step-size	341
1. The idea of Richardson (341). 2. The Chebyshev	
set of parameters (342). 3. Numbering of iteration	
parameters (346). 4. The Douglas-Rachford method (349).	
Problems	352
§ 37. The Federenko method	353
1. Idea of the method (354). 2. Description of the	
algorithm (356).	

Chapter 12. Concept of Variational-Difference and Projection-Difference Schemes	357
§ 38. Variational and projection methods	357
1. Variational formulation of boundary-value problems (357). 2. Convergence of minimizing sequences (361). 3. The variational method of Ritz (365). 4. Projection method of Galerkin (371). 5. Methods for solving the algebraic system (373). 6. Computational stability (373). Problems	374
§ 39. Construction and properties of variational-difference and projection-difference schemes	375
1. Definition of variational-difference and projection-difference schemes (375). 2. Example of a variational-difference scheme for the first boundary-value problem (376). 3. An example of a variational-difference scheme for the third boundary-value problem (385). 4. On the method for proving convergence (388). 5. Comparison of variational-difference schemes with general variational and ordinary difference schemes (389). Problems	390

PART 5

**STABILITY OF EVOLUTIONAL BOUNDARY-VALUE PROBLEMS VIEWED
AS THE BOUNDEDNESS OF NORMS OF POWERS OF A CERTAIN OPERATOR**

Chapter 13. Construction of the Transition Operator	392
§ 40. Level structure of the solution of evolutionary problems	392
Problems	395
§ 41. Statement of the difference boundary-value problem in the form $u^{p+1} = R_h u^p + \tau \rho^p$	396
1. Canonical form (396). 2. Stability as the uniform boundedness of the norms of powers of R_h (400). 3. Example (403). Problems	406
§ 42. Use of particular solutions in the construction of the transition operator	408

§ 43. Some methods for bounding norms of powers of operators	421
1. Necessary spectral conditions for the boundedness of $\ R_h^p\ $ (422).	
2. Spectral criterion for the boundedness of powers of a selfadjoint operator (424).	
3. Self-adjointness criteria (425).	
4. Bounds on the eigenvalues of operator R_h (426).	
5. Choice of a scalar product (428).	
6. The stability criterion of Samarskii (429).	
Problems	431
Chapter 14. Spectral Criterion for the Stability of Nonselfadjoint Evolutionary Boundary-Value Problems	433
§ 44. Spectrum of a family of operators $\{R_h\}$	433
1. Need for improvement in the spectral stability criterion (433).	
2. Definition of the spectrum of a family of operators (435).	
3. Necessary condition for stability (436).	
4. Discussion of the concept of the spectrum of a family of operators $\{R_h\}$ (437).	
5. Nearness of the necessary stability criterion to sufficiency (439).	
§ 45. Algorithm for the computation of the spectrum of a family of difference operators on net functions in an interval	441
1. Typical example (441).	
2. Algorithm for computing the spectrum in the general case (449).	
Problems	450
§ 46. The kernels of the spectra of families of operators	451
§ 47. On the stability of iterative algorithms for the solution of nonselfadjoint difference equations	455
Appendix. Method of internal boundary conditions	461
1. Class of systems of difference equations (461).	
2. Fundamental solution (462).	
3. Boundary of net-region (463).	
4. Difference analogs of Cauchy and Cauchy-type integral formulas (464).	
5. Internal boundary conditions (466).	
6. Boundary projection operator (466).	
7. General boundary-value problem (467).	
8. Basic idea of the method of internal boundary conditions (467).	
9. Stability of internal boundary conditions (468).	
10. Supplementary idea (469).	
11. Comparison of the method of internal boundary conditions with the method of singular integral equations (470).	
Bibliographical commentaries	475
Bibliography	483
Index	485