

# Contents

<b>ALGEBRAIC THEORY</b>	<b>1</b>
<b>1 Picard-Vessiot rings</b>	<b>4</b>
1.1 Existence and uniqueness of Picard-Vessiot rings . . . . .	7
1.2 The Galois group . . . . .	8
1.3 Galois correspondence for difference equations . . . . .	16
1.4 Difference modules and fibre functors . . . . .	23
<b>2 Algorithms for difference equations</b>	<b>28</b>
2.1 Difference equations of order one . . . . .	28
2.2 Difference equations in diagonal form . . . . .	31
2.3 Difference equations of order two . . . . .	33
<b>3 The inverse problem for difference equations</b>	<b>35</b>
<b>4 The ring <math>\mathcal{S}</math> of sequences</b>	<b>45</b>
<b>5 An excursion in positive characteristic</b>	<b>52</b>
5.1 Generalities . . . . .	52
5.2 Modules over $K[T, T^{-1}]$ . . . . .	56
5.3 Difference Galois groups . . . . .	57
5.4 Comparing characteristic 0 and $p$ . . . . .	58
<b>6 Difference modules over <math>\mathcal{P}</math></b>	<b>60</b>
6.1 Classification of difference modules over $\mathcal{P}$ . . . . .	60
6.2 The universal Picard-Vessiot ring of $\mathcal{P}$ . . . . .	63
6.3 Fields of constants which are not algebraically closed . . . . .	65

6.4	Automorphisms of the universal Picard-Vessiot ring of $\mathcal{P}$ . . . . .	65
6.5	Difference equations over $C((z^{-1}))$ and the formal Galois group. . . . .	66
<b>ANALYTIC THEORY</b>		<b>68</b>
<b>7</b>	<b>Classification and canonical forms</b>	<b>71</b>
7.1	A classification of singularities . . . . .	71
7.2	Canonical forms . . . . .	75
<b>8</b>	<b>Semi-regular difference equations</b>	<b>77</b>
8.1	Introduction . . . . .	77
8.2	Some easy asymptotics . . . . .	78
8.3	The connection matrix of a semi-regular equation . . . . .	80
8.4	The theorem of Malgrange and Sibuya . . . . .	84
8.5	Regular difference equations . . . . .	86
8.6	Inverse problems for semi-regular equations . . . . .	88
<b>9</b>	<b>Mild difference equations</b>	<b>95</b>
9.1	Asymptotics for mild equations . . . . .	95
9.2	Connection matrices of mild equations . . . . .	96
9.3	Tame differential modules . . . . .	105
9.4	Inverse problems for mild equations . . . . .	106
<b>10</b>	<b>Examples of equations and Galois groups</b>	<b>111</b>
10.1	Calculating connection matrices . . . . .	111
10.2	Classification of order one equations . . . . .	116
10.3	More on difference Galois groups . . . . .	119
10.4	Mild difference and differential equations . . . . .	122
10.5	Very mild difference modules and multisummability . . . . .	124
10.6	Very mild differential modules . . . . .	125
<b>11</b>	<b>Wild difference equations</b>	<b>127</b>
11.1	Introduction . . . . .	127
11.2	Multisummability of formal solutions . . . . .	128
11.3	The Quadrant Theorem . . . . .	129
11.4	On the Gamma function . . . . .	130

11.5	An example . . . . .	131
11.6	Solutions on a right half plane . . . . .	133
11.7	Solutions on an upper half plane . . . . .	137
11.8	Analytic equivalence classes of difference equations . . . . .	140
11.9	An example . . . . .	144
<b>12</b>	<b><math>q</math>-difference equations</b>	<b>149</b>
12.1	Formal aspects . . . . .	149
12.2	Analytic properties . . . . .	153
12.2.1	Regular singular equations over $k_0$ . . . . .	154
12.2.2	Equations over $\mathbf{C}(z)$ . . . . .	156
12.3	Construction of the connection map . . . . .	157
12.3.1	Meromorphic vector bundles . . . . .	157
12.3.2	The connection map of a regular equation . . . . .	159
12.3.3	The connection map of a regular singular equation . . . . .	162
12.3.4	Inverse problems . . . . .	166
	<b>Bibliography</b>	<b>175</b>
	<b>Index</b>	<b>179</b>
	<b>Notations</b>	<b>180</b>