

## Contents

---

DEDICATION . . . . .	v
INTRODUCTION . . . . .	vii

### Chapter 1. The Laplace Transform

1.1. Introduction . . . . .	1
1.2. Existence and Convergence <i>Exercises</i> . . . . .	1
1.3. The Inversion Problem . . . . .	3
1.4. Behavior of the Dirichlet Kernel . . . . .	4
1.5. Analytic Details <i>Exercises</i> . . . . .	4
1.6. Statement of Result . . . . .	7
1.7. Jump Discontinuity . . . . .	8
1.8. Functions of Bounded Variation . . . . .	8
1.9. Contour Integration . . . . .	9
1.10. Examples . . . . .	10
1.11. The Fejér Transform . . . . .	11
1.12. The Inverse Inversion Problem . . . . .	12
1.13. The Convolution Theorem <i>Exercises</i> . . . . .	13
1.14. The Fourier Transform . . . . .	17
1.15. Plancherel-Parseval Theorem . . . . .	18
1.16. Application to Laplace Transform . . . . .	19
1.17. The Post-Widder Formula . . . . .	19
1.18. Real Inversion Formulas <i>Exercise</i> . . . . .	20
Miscellaneous Exercises and Research Problems . . . . .	21
Bibliography and Comments . . . . .	26

### Chapter 2. Linear Differential Equations

2.1. Introduction . . . . .	27
2.2. Linear Differential Equations . . . . .	27
2.3. Fundamental Existence and Uniqueness Theorem . . . . .	29
2.4. Successive Approximations . . . . .	29
2.5. A Fundamental Lemma . . . . .	31
2.6. Uniqueness Theorem . . . . .	31
2.7. Fixed-point Techniques . . . . .	32
2.8. Difference Schemes . . . . .	32
2.9. The Matrix Equation . . . . .	33
2.10. Alternative Derivation . . . . .	33
2.11. The Inhomogeneous Equation . . . . .	34
2.12. The Adjoint Equation . . . . .	34
2.13. Constant Coefficients—I . . . . .	35

2.14. Constant Coefficients—II . . . . .	36
2.15. Laplace Transform Solution . . . . .	36
2.16. Characteristic Values and Characteristic Functions . . . . .	37
Miscellaneous Exercises and Research Problems . . . . .	40
Bibliography and Comments . . . . .	41

### **Chapter 3. First-order Linear Differential-Difference Equations of Retarded Type with Constant Coefficients**

3.1. Introduction . . . . .	42
3.2. Examples <i>Exercises</i> . . . . .	45
3.3. Equations of Retarded, Neutral, and Advanced Type . . . . .	48
3.4. The Existence-Uniqueness Theorem <i>Exercises</i> . . . . .	49
3.5. Exponential Solutions <i>Exercises</i> . . . . .	53
3.6. Order of Growth of Solutions <i>Exercises</i> . . . . .	58
3.7. Laplace Transform Solution <i>Exercises</i> . . . . .	63
3.8. Solution of a Differential Equation in the Form of a Definite Integral . . . . .	71
3.9. Solution of a Differential-Difference Equation in the Form of a Definite Integral <i>Exercises</i> . . . . .	73
Miscellaneous Exercises and Research Problems . . . . .	80
Bibliography and Comments . . . . .	97

### **Chapter 4. Series Expansions of Solutions of First-order Equations of Retarded Type**

4.1. The Characteristic Roots . . . . .	98
4.2. Series Expansions . . . . .	102
4.3. Other Forms of the Expansion Theorem <i>Exercises</i> . . . . .	108
4.4. Asymptotic Behavior of the Solution <i>Exercises</i> . . . . .	113
4.5. Stability of Equilibrium <i>Exercises</i> . . . . .	117
4.6. Fourier-type Expansions <i>Exercises</i> . . . . .	121
4.7. The Shift Theorem <i>Exercises</i> . . . . .	126
Miscellaneous Exercises and Research Problems . . . . .	131
Bibliography and Comments . . . . .	137

### **Chapter 5. First-order Linear Equations of Neutral and Advanced Type with Constant Coefficients**

5.1. Existence-Uniqueness Theorems <i>Exercises</i> . . . . .	139
5.2. Solution by Exponentials and by Definite Integrals: Equations of Neutral Type <i>Exercises</i> . . . . .	143
5.3. Series Expansions: Equations of Neutral Type <i>Exercise</i> . . . . .	153
5.4. Asymptotic Behavior and Stability: Equations of Neutral Type . . . . .	158
5.5. Other Expansions for Solutions of Equations of Neutral Type <i>Exercises</i> . . . . .	159
5.6. Equations of Advanced Type . . . . .	160
Miscellaneous Exercises and Research Problems . . . . .	161

## Chapter 6. Linear Systems of Differential-Difference Equations with Constant Coefficients

6.1.	Introduction.....	164
6.2.	Vector-matrix Notation.....	165
6.3.	Classification of Systems.....	165
6.4.	Existence-Uniqueness Theorems for Systems <i>Exercises</i> .....	167
6.5.	Transform Solutions: Retarded-Neutral Systems <i>Exercises</i> .....	173
6.6.	Solution of Neutral and Retarded Systems by Definite Integrals <i>Exercises</i> .....	179
6.7.	Series Expansions for Neutral and Retarded Systems <i>Exercises</i> .....	183
6.8.	Asymptotic Behavior of Solutions of Neutral and Retarded Systems <i>Exercises</i> .....	188
6.9.	Scalar Equations <i>Exercises</i> .....	192
6.10.	The Finite Transform Method <i>Exercise</i> .....	197
6.11.	Fourier-type Expansions <i>Exercises</i> .....	206
	Miscellaneous Exercises and Research Problems.....	208
	Bibliography and Comments.....	214

## Chapter 7. The Renewal Equation

7.1.	Introduction.....	216
7.2.	Existence and Uniqueness <i>Exercises</i> .....	217
7.3.	Further Existence and Uniqueness Theorems.....	220
7.4.	Monotonicity and Bounded Variation <i>Exercises</i> .....	224
7.5.	The Formal Laplace Transform Solution.....	226
7.6.	Exponential Bounds for $u(t)$ .....	227
7.7.	Rigorous Solution.....	228
7.8.	A Convolution Theorem.....	229
7.9.	Asymptotic Behavior of Solutions.....	231
7.10.	Use of the Contour Integral Representation.....	231
7.11.	$\phi(t)$ a Positive Function.....	232
7.12.	Shift of the Contour.....	233
7.13.	Step Functions.....	234
7.14.	An Elementary Result.....	236
7.15.	A Less Easily Obtained Result.....	236
7.16.	Abelian and Tauberian Results.....	239
7.17.	A Tauberian Theorem of Hardy and Littlewood.....	240
7.18.	Asymptotic Behavior of Solution of Renewal Equation.....	240
7.19.	Discussion.....	241
7.20.	A Tauberian Theorem of Ikehara.....	242
7.21.	The Tauberian Theorem of Wiener.....	243
	Miscellaneous Exercises and Research Problems.....	244
	Bibliography and Comments.....	255

## Chapter 8. Systems of Renewal Equations

8.1.	Introduction.....	257
8.2.	Vector Renewal Equation.....	257

8.3.	Positive Matrices . . . . .	258
8.4.	Some Consequences . . . . .	259
8.5.	Zero with Largest Real Part . . . . .	259
8.6.	Asymptotic Behavior . . . . .	262
	Miscellaneous Exercises and Research Problems . . . . .	262
	Bibliography and Comments . . . . .	263

### Chapter 9. Asymptotic Behavior of Linear Differential-Difference Equations

9.1.	Introduction . . . . .	265
9.2.	First Principal Result . . . . .	266
9.3.	Preliminaries . . . . .	267
9.4.	Discussion . . . . .	268
9.5.	$\int^{\infty}  a(t_1)  dt_1 < \infty$ . . . . .	268
9.6.	The Difficult Part of Theorem 9.1 . . . . .	270
9.7.	A Lemma . . . . .	272
9.8.	Continuation of Proof of Theorem 9.1 . . . . .	274
9.9.	The Case Where $b(t) \neq 0$ . . . . .	275
9.10.	Further Results . . . . .	278
9.11.	More Precise Results . . . . .	278
9.12.	Asymptotic Series . . . . .	278
9.13.	The Foundations of Asymptotic Series . . . . .	280
9.14.	Alternative Formulation . . . . .	282
9.15.	Differential and Integral Properties . . . . .	282
9.16.	Extension of Definition <i>Exercises</i> . . . . .	283
9.17.	First-order Linear Differential Equations . . . . .	284
9.18.	Second-order Linear Differential Equations . . . . .	285
9.19.	The Case Where $a_0 = 0$ <i>Exercise</i> . . . . .	286
9.20.	A Rigorous Derivation of the Asymptotic Expansion . . . . .	287
9.21.	Determination of the Constants <i>Exercises</i> . . . . .	289
9.22.	A Basic Problem in the Theory of Differential Equations . . . . .	289
9.23.	Formal Determination of Coefficients . . . . .	290
9.24.	Asymptotic Expansion of Solution . . . . .	291
	Miscellaneous Exercises and Research Problems . . . . .	292
	Bibliography and Comments . . . . .	299

### Chapter 10. Stability of Solutions of Linear Differential-Difference Equations

10.1.	Introduction . . . . .	300
10.2.	Stability Theory for Ordinary Differential Equations . . . . .	300
10.3.	The Adjoint Equation . . . . .	302
10.4.	The Scalar Linear Differential-Difference Equation . . . . .	304
10.5.	The Matrix Equation with Retarded Argument . . . . .	306
10.6.	A Stability Theorem for Equations with Retarded Argument . . . . .	308
10.7.	Equations with Constant Coefficients . . . . .	310
10.8.	A Lemma . . . . .	311
10.9.	A Stability Theorem for Equations with Constant Coefficients . . . . .	312
10.10.	Boundedness of Solutions of the Unperturbed System . . . . .	313
10.11.	The Scalar Equation of Neutral Type: Integral Representation for a Solution <i>Exercise</i> . . . . .	313

10.12. The Scalar Equation of Neutral Type: Representation for the Derivative of a Solution.....	316
10.13. Systems of Equations of Neutral Type.....	320
10.14. Stability Theorems for Equations of Neutral Type.....	324
10.15. Stability Theorems for Equations of Neutral Type with Constant Coefficients.....	327
Miscellaneous Exercises and Research Problems.....	330
Bibliography and Comments.....	333

### Chapter 11. Stability Theory and Asymptotic Behavior for Nonlinear Differential-Difference Equations

11.1. Introduction.....	334
11.2. The Poincaré-Liapunov Theorem.....	335
11.3. Small Perturbations for General Systems.....	337
11.4. Types of Stability.....	339
11.5. Existence Theorem for Nonlinear Differential-Difference Equations.....	341
11.6. Uniqueness.....	344
11.7. Statement of Existence and Uniqueness Theorems <i>Exercises</i> .....	345
11.8. Stability Theorem <i>Exercise</i> .....	348
11.9. Stability Theorem: Second Proof <i>Exercise</i> .....	350
11.10. Asymptotic Behavior of the Solutions.....	354
11.11. Proof of Theorem 11.5 <i>Exercises</i> .....	356
11.12. Another Stability Theorem.....	361
11.13. Dini-Hukuhara Theorem for Equations with Variable Coefficients.....	364
11.14. Poincaré-Liapunov Theorem for Equations with General Variable Coefficients.....	367
11.15. Asymptotic Behavior for Nonlinear Equations with Almost-constant Coefficients.....	369
11.16. Systems of Nonlinear Equations.....	372
11.17. Liapunov Functions and Functionals.....	373
Miscellaneous Exercises and Research Problems.....	376
Bibliography and Comments.....	390

### Chapter 12. Asymptotic Location of the Zeros of Exponential Polynomials

12.1. Introduction.....	393
12.2. The Form of $\det H(s)$ .....	394
12.3. Zeros of Analytic Functions <i>Exercises</i> .....	395
12.4. Constant Coefficients and Commensurable Exponents.....	399
12.5. Constant Coefficients and General Real Exponents.....	400
12.6. Asymptotically Constant Coefficients.....	404
12.7. Polynomial Coefficients with $m_j$ and $\beta_j$ Proportional.....	406
12.8. Polynomial Coefficients.....	410
12.9. Examples <i>Exercise</i> .....	416
12.10. Conditions That All Roots Be of Specified Type <i>Exercises</i> .....	417
12.11. Construction of Contours.....	420

12.12. Order Results for $H^{-1}(s)$ . . . . .	422
12.13. Order Results in the Scalar Case . . . . .	423
12.14. Convergence of Integrals over the Contours <i>Exercises</i> . . . . .	424
12.15. Integrals along Vertical Lines . . . . .	427
Miscellaneous Exercises and Research Problems . . . . .	432
Bibliography and Comments . . . . .	439

### Chapter 13. On Stability Properties of the Zeros of Exponential Polynomials

13.1. Introduction . . . . .	440
13.2. Exponential Polynomials <i>Exercises</i> . . . . .	440
13.3. Functions of the Form $f(z, \cos z, \sin z)$ . . . . .	441
13.4. Presence of a Principal Term . . . . .	442
13.5. Zeros of $h(z, e^z)$ . . . . .	442
13.6. The Fundamental Stability Results . . . . .	443
13.7. A Result of Hayes . . . . .	444
13.8. An Important Equation . . . . .	446
13.9. Another Example . . . . .	450
Miscellaneous Exercises and Research Problems . . . . .	452
Bibliography and Comments . . . . .	454
 AUTHOR INDEX . . . . .	457
 SUBJECT INDEX . . . . .	460
 OTHER RAND Books . . . . .	463