

# Contents

Preface		vii
1	The Direct Method of the Calculus of Variations	1
	Introduction	1
1.1	Lower Semi-Continuous Functions	3
1.2	Convex Functions	4
1.3	Euler Equation	5
1.4	The Calculus of Variations with Periodic Boundary Conditions	6
1.5	Periodic Solutions of Non-Autonomous Second Order Systems with Bounded Nonlinearity	12
1.6	Periodic Solutions of Non-Autonomous Second Order Systems with Periodic Potential	14
1.7	Periodic Solutions of Non-Autonomous Second Order Systems with Convex Potential	17
	Historical and Bibliographical Notes	22
	Exercises	25
2	The Fenchel Transform and Duality	28
	Introduction	28
2.1	Definition of the Fenchel Transform	30
2.2	Differentiable Convex Functions	34
2.3	Hamiltonian Duality	35
2.4	Clarke Duality	37
	Historical and Bibliographical Notes	39
	Exercises	40
3	Minimization of the Dual Action	42
	Introduction	42
3.1	Eigenvalues of Eigenfunctions of $J(d/dt)$ with Periodic Boundary Conditions	43

3.2	A Basic Existence Theorem for Periodic Solutions of Convex Hamiltonian Systems	45
3.3	Subharmonics of Non-Autonomous Convex Hamiltonian Systems	50
3.4	Periodic Solutions with Prescribed Minimal Period of Autonomous Convex Hamiltonian Systems	53
3.5	Periodic Solutions with Prescribed Energy of Autonomous Hamiltonian Systems	56
3.6	Periodic Solutions of Non-Autonomous Second Order Systems with Convex Potential	60
3.7	A Variant of the Dual Least Action Principle for Non-Autonomous Second Order Systems	61
3.8	The Range of Some Second Order Nonlinear Operators with Periodic Boundary Conditions	67
	Historical and Bibliographical Notes	70
	Exercises	71
4	Minimax Theorems for Indefinite Functionals	73
	Introduction	73
4.1	Ekeland's Variational Principle and the Existence of Almost Critical Points	75
4.2	A Closedness Condition and the Existence of Critical Points	81
4.3	The Saddle Point Theorem and Periodic Solutions of Second Order Systems with Bounded Nonlinearity	85
4.4	Periodic Solutions of Josephson-Type Systems	87
4.5	The Mountain Pass Theorem and Periodic Solutions of Superlinear Convex Autonomous Hamiltonian Systems	92
4.6	Multiple Critical Points of Periodic Functionals	97
	Historical and Bibliographical Notes	106
	Exercises	107
5	A Borsuk-Ulam Theorem and Index Theories	111
	Introduction	111
5.1	Group Representations	112
5.2	The Parametrized Sard Theorem	114
5.3	Topological Degree	116
5.4	Index Theories	120
	Historical and Bibliographical Notes	123
	Exercises	124

6	Lusternik–Schnirelman Theory and Multiple Periodic Solutions with Fixed Energy	126
	Introduction	126
6.1	Equivariant Deformations	127
6.2	Existence of Multiple Critical Points	132
6.3	Multiple Periodic Solutions with Prescribed Energy of Autonomous Hamiltonian Systems	134
6.4	Nonlinear Eigenvalue Problems	139
6.5	Application to Bifurcation Theory	141
6.6	Multiple Periodic Solutions with Prescribed Energy Near an Equilibrium	148
	Historical and Bibliographical Notes	149
	Exercises	151
7	Morse–Ekeland Index and Multiple Periodic Solutions with Fixed Period	153
	Introduction	153
7.1	The Index of a Linear Positive Definite Hamiltonian System	154
7.2	Linear Autonomous Positive Definite Hamiltonian Systems	160
7.3	Periodic Solutions of Convex Asymptotically Linear Autonomous Hamiltonian Systems	161
	Historical and Bibliographical Notes	165
	Exercises	165
8	Morse Theory	167
	Introduction	167
8.1	Relative Homology	168
8.2	Manifolds	173
8.3	Vector Fields	175
8.4	Riemannian Manifolds	176
8.5	Morse Inequalities	179
8.6	The Generalized Morse Lemma	184
8.7	Computation of the Critical Groups	189
8.8	Critical Groups at a Point of Mountain Pass Type	195
8.9	Continuity of the Critical Groups and Bifurcation Theory	196
8.10	Lower Semi-Continuity of the Betti Numbers	199
8.11	Critical Groups at a Saddle Point	200

	Historical and Bibliographical Notes	201
	Exercises	203
9	Applications of Morse Theory to Second Order Systems	205
	Introduction	205
9.1	The Index of a Linear Second Order Differential System	206
9.2	Periodic Solutions of Autonomous Second Order Systems Near an Equilibrium	207
9.3	Periodic Solutions of Asymptotically Linear Non-Autonomous Second Order Systems	210
9.4	Multiple Solutions of Lagrangian Systems	214
	Historical and Bibliographical Notes	215
	Exercises	216
10	Nondegenerate Critical Manifolds	217
	Introduction	217
10.1	Submanifolds	217
10.2	Normal Bundle	219
10.3	Critical Groups of a Nondegenerate Critical Manifold	221
10.4	Global Theory	224
10.5	Second Order Autonomous Superlinear Equations	227
10.6	Periodic Solutions of Forced Superlinear Second Order Equations	231
10.7	Local Perturbations of Nondegenerate Critical Manifolds	235
	Historical and Bibliographical Notes	238
	Exercises	238
	Bibliography	240
	Index	275