Table of Contents

Preface

Acknowledgments & Dedications

Chapter 0 Introduction 1

0.0 Motivation 1

0.1 Literature Review 6

0.2 Preview 10

Chapter 1 Modelling Diffusion, Drift and Boundary Conditions 13

1.0 On Mathematical Modelling 13

A) Introduction 13

B) Background 15

1.1 The Diffusion-Random Walk Connection 18

1.2 The Random Walk-Brownian Motion Connection 21

1.3 Diffusion and Markovian Processes 24

1.4 The Drift Phenomenon 25

1.5 The Lagrangian Viewpoint 26

1.6 The Eulerian Viewpoint 39

1.7 Diffusion, Drift, and the Divergence Form 42

1.8 Modelling the Boundary Conditions 43

Chapter 2 Existence, Coercivity and Monotonicity Results 51

2.0 Introduction 51

2.1 The Adjoint Problem 52

2.2 Existence of the Principal Eigenvalue 55

A) The Dirichlet case (BC1) 56

B) The No-flux (Neumann) and Robin cases (BC2, BC3) 57

2.3 Coercivity Conditions 68

A) Coercivity for BC1 and BC3 71

B) Coercivity for BC2 76

2.4 Some Monotonicity Results 78

Chapter 3 The Case of Potential Based Drift 81

3.0 Divergence Free and Conservative Vector Fields 82

3.1 Principal Eigenvalue Characterizations 84

3.2 Bounds for the Principal Eigenvalue 88

A) Extensions to the Murray-Sperb bounds 89

B) The Protter scheme 91

C) Other inequalities 96

3.3 The Role of Potential-based Drift in Persistence 99

A) The Lazer argument 100

B) Computation of $\frac{\partial \lambda}{\partial \alpha}$. 101

C) The Neumann case and the Cosner conjecture 107

D) Computation of $\frac{\partial^2 \lambda}{\partial \alpha^2}$ 108

Chapter 4 Degenerate Elliptic Equations 111

4.0 Background 111

4.1 Known Analytic Results 113

4.2 The One-Dimensional Case 118

4.3 Preliminaries to the Probabilistic View Point 123

4.4 Recurrence, Transcience and Hypoellipticity 134

4.5 On a Degenerate Elliptic Equation 143

A) The one-dimensional case ($n = 1$) 144

B) The Potential based drift case ($b = -a \bigtriangledown F$) 144

C) Arbitrary (ba^{-1}), smooth coefficients a, b and m 144

D) Summary 146

Chapter 5 Minimax Formulations of the Principal Eigenvalue for Indefinite Weights 149

5.0 Background 149

5.1 Minimax Formulation for λ_N^* 154

5.2 Minimax Formulation for λ_R^* 162

5.3 Minimax Formulation for λ_D^* 164

5.4 Alternative Forms of the Characterizations 169

5.5 Interchange of Min and Max in the Formulations 174

Epilogue 177

Appendix Preliminaries 179

A) Basics 179

B) Arrays and matrices 181

C) Second-order linear differential operators and equations 183

D) Maximum principles 188

E) Perron-Frobenius and Krein-Rutman theorems 189

F) Minimax theorems 191

G) Measure, outer measure and measure spaces 195

H) Distributions and hypoelliptic operators 202

Bibliography 207