Contents

Preface v

I Basic Framework 1

1 The Physical Context 3
1.1 Weakly Nonlinear Dispersive Waves 3
1.1.1 A Weakly Nonlinear Dispersion Relation 3
1.1.2 Derivation in Terms of Fourier-Mode Coupling . . 6
1.1.3 Introduction to Multiple-Scale Analysis 7
1.2 The Example of Optical Waves 10
1.2.1 Waves in a Dielectric 11
1.2.2 The Paraxial Approximation in a Linear Medium . 11
1.2.3 Static Modulation in a Kerr Medium 12
1.2.4 Time Dispersion of Ultrashort Wave Trains 14
1.3 Basic Dynamical Effects 18
1.3.1 Modulational Instability 18
1.3.2 Solitons in One Space Dimension 20
1.3.3 Soliton Instability for Transverse Perturbations . . 22
1.4 Fluid-Dynamical Form of the NLS Equation 25

2 Structural Properties 27
2.1 Variational Formulation 27
2.1.1 Lagrangian Structure 27
2.1.2 Hamiltonian Structure 29

2.2 The Noether Theorem 30
2.3 Invariances and Conservation Laws 33
2.4 Variance Identity 37
2.4.1 Elliptic NLS Equation 38
2.4.2 Extension to the Non-Elliptic NLS Equation . . . 39

II Rigorous Theory 41

3 Existence and Long-Time Behavior 43
3.1 The Linear Schrödinger Operator 44
3.1.1 Basic Estimates 45
3.1.2 Linear Schrödinger Equation with Potential 48
3.1.3 Smoothing Properties 49
3.2 Existence Properties 51
3.2.1 Local Existence 51
3.2.2 Global Existence for Large Initial Conditions . . . 54
3.2.3 Global Regularity for Small Initial Conditions . . . 55
3.2.4 Self-Similar Solutions 57
3.3 Scattering Properties 59
3.3.1 The Case of Repulsive Nonlinearity 61
3.3.2 The case of Attracting Nonlinearity 63
3.4 Further Results 64
3.4.1 Generalized NLS Equations 64
3.4.2 Defocusing NLS with Nonzero Condition at Infinity 66
3.4.3 Periodic Boundary Conditions and Invariant Measures 68

4 Standing Wave Solutions 71
4.1 A Heuristic Approach 72
4.1.1 Linear Stability Analysis 72
4.1.2 The Case of Finite Amplitude Perturbations 75
4.2 Existence and Variational Approach 76
4.2.1 Necessary Conditions for Existence in H^1 76
4.2.2 Existence Results 77
4.2.3 Variational Approaches 78
4.3 Stability/Instability Conditions 80
4.3.1 Subcritical Dimension: Orbital Stability 81
4.3.2 Critical Case: Instability by Blowup 83
4.3.3 Supercritical Case: Instability by Blowup 84
4.4 A General Approach for Hamiltonian Systems 85
4.4.1 Setting of the Problem and Main Results 85
4.4.2 Main Steps of the Proofs 87
4.4.3 Extension to Abstract Hamiltonian Systems 90

4.5 Further Stability Results 91
4.5.1 Asymptotic Stability Results 91
4.5.2 Ground-State Orbital Stability and Global Existence 92

5 Blowup Solutions **93**
5.1 Finite-Time Blowup 93
5.1.1 Case of Finite Variance 93
5.1.2 Extensions to Solutions with Infinite Variance . . . 98
5.2 Analysis of the Blowup 103
5.2.1 Rate of Blowup 103
5.2.2 A Self-Similar Solution at Critical Dimension . . . 104
5.2.3 Solutions with Exactly k Blowup Points 106
5.2.4 L^2-Norm Concentration at Critical Dimension . . . 106

III Asymptotic Analysis near Collapse **113**

6 Numerical Observations **115**
6.1 Capturing the Blowup Structure 117
6.1.1 A Scale Transformation for Isotropic Solutions . . 117
6.1.2 Dynamic Rescaling 118
6.2 Simulation of Isotropic Collapse 120
6.2.1 Stability of Supercritical Self-Similar Solutions . . 120
6.2.2 The NLS Equation at Critical Dimension 121
6.2.3 An Adaptive Galerkin Finite Element Method . . 123
6.3 Simulation of Non-Radially Symmetric Solutions 126
6.3.1 Anisotropic Dynamic Rescaling 126
6.3.2 Stability of Isotropic Collapse 130
6.3.3 An Iterative Grid Redistribution Method 131

7 Supercritical Collapse **133**
7.1 Self-Similar Blowup Solutions 133
7.1.1 Properties of the Profile 134
7.1.2 Spatial Extension of the Self-Similar Profile 136
7.1.3 Rate of Convergence to Self-Similar Solutions . . . 137
7.2 Dissipation and Postcollapse Dynamics 138

8 Critical Collapse **141**
8.1 Self-Similar Profile near Critical Dimension 142
8.1.1 Constraints on the Self-Similar Profile 142
8.1.2 A Nonlinear Eigenvalue Problem 142
8.1.3 A Nonuniform Limit 147
8.1.4 Remarks on the Critical Profile 150
8.2 Asymptotic Solutions at Critical Dimension 151

8.2.1 Construction of Asymptotic Solutions 151
8.2.2 Effects of Mass and Hamiltonian Radiation 155
8.2.3 Adiabatic Approximation and Blowup Time Estimate 158
8.2.4 Relation with Standing-Wave Instability 159

9 Perturbations of Focusing NLS **161**
9.1 Wave Dissipation at Critical Dimension 161
9.1.1 Effect of an Individual Collapse 161
9.1.2 The Turbulent Regime 164
9.2 Non-Elliptic Schrödinger Equation 164
9.2.1 Estimates for the Partial Variances 165
9.2.2 Numerical Observations in Two Dimensions 168
9.2.3 Numerical Simulations in Three Dimensions 169
9.2.4 Effect of a Small Time Dispersion on Critical Collapse 170
9.3 Saturated Nonlinearity 173
9.3.1 Standing-Wave Solutions 175
9.3.2 Numerical Observations 176
9.3.3 Saturation of Critical Collapse 179
9.4 Other Pertubations 183
9.4.1 Weak Nonparaxiality 183
9.4.2 General Formalism 185

IV Coupling to a Mean Field **187**

10 Mean Field Generation **189**
10.1 General Formalism 189
10.2 A few simple examples 194
10.2.1 The Korteweg–de Vries equation 194
10.2.2 The Boussinesq equation 196
10.2.3 The Kadomtsev–Petviashvili equation 199
10.3 A Degenerate Case 201

11 Gravity-Capillary Surface Waves **205**
11.1 The Water-Wave Problem 206
11.1.1 Equations Governing the Interface Motion 206
11.1.2 Formal Modulation Analysis 207
11.2 Error Bounds 216
11.2.1 Preliminaries 216
11.2.2 Modulation of the Water-Wave Operator 217

12 The Davey–Stewartson System **221**
12.1 General Setting 221

12.1.1 Boundary Conditions 221
12.1.2 Expression of the Mean Flow 222
12.1.3 Conservation Properties 223
12.2 Standing-Wave Solutions 226
12.2.1 The Elliptic–Elliptic Case 226
12.2.2 The Hyperbolic–Elliptic case 228
12.3 The Initial Value Problem 230
12.3.1 Subsonic Wave Packet 230
12.3.2 Supersonic Wave Packets 232
12.4 Rate of Blowup for Elliptic–Elliptic DS 234
12.5 Solutions of Elliptic–Hyperbolic DS 240

V Coupling to Acoustic Waves 243

13 Langmuir Oscillations 245
13.1 Derivation of the Zakharov Equations 246
13.1.1 The Two-Fluid Model 246
13.1.2 The Vector Zakharov Equations 247
13.1.3 The Electrostatic Limit 250
13.1.4 Generation of a Large-Scale Magnetic Field 252
13.2 Rigorous Results 254
13.2.1 Existence Theory 254
13.2.2 The Subsonic Limit 256
13.2.3 The Vector NLS Equation 256
13.3 Evidence of Collapse 258
13.3.1 Heuristic Arguments 258
13.3.2 Simulations in the Electrostatic Approximation . . 259
13.3.3 Simulations of the Vector Equations 260

14 The Scalar Model 263
14.1 Self-Similar Solutions 264
14.1.1 Formal Construction 264
14.1.2 Dynamical Stability 267
14.2 Existence and Blowup Results 270
14.2.1 Existence Theory 270
14.2.2 Blowup Results 274
14.3 Further Analysis in Two Dimensions 278
14.3.1 Existence of Self-Similar Blowup Solutions 278
14.3.2 Mass Concentration Properties 280
14.3.3 A Sharp Existence Result 281
14.3.4 Instability of Standing Wave Solutions 283
14.3.5 An Optimal Lower Bound for the Blowup Rate . . 283

15 Progressive Waves in Plasmas **287**
15.1 Interaction with a Nonmagnetic Medium 287
15.1.1 Laser Beams in Plasmas 287
15.1.2 Hamiltonian Formalism 288
15.2 Alfvén Wave Filamentation 291
15.2.1 Modulation Equations 291
15.2.2 Influence of the Magnetosonic Waves 298
15.3 Weakly Dispersive Alfvén Waves 303
15.3.1 Parallel Propagation 303
15.3.2 Oblique Propagation 306

Synopsis **307**

References **309**

Name Index **339**

Subject Index **347**