

Contents

Preface	ix
Acknowledgments	xi
1 Introduction	1
1.1 The first order case	1
1.1.1 Statement of the problem	1
1.1.2 The scalar case	2
1.1.3 Some examples in the vectorial case	4
1.1.4 Convexity conditions in the vectorial case	8
1.1.5 Some typical existence theorems in the vectorial case	9
1.2 Second and higher order cases	10
1.2.1 Dirichlet-Neumann boundary value problem	10
1.2.2 Fully nonlinear partial differential equations	12
1.2.3 Singular values	13
1.2.4 Some extensions	14
1.3 Different methods	15
1.3.1 Viscosity solutions	15
1.3.2 Convex integration	17
1.3.3 The Baire category method	18
1.4 Applications to the calculus of variations	20
1.4.1 Some bibliographical notes	21
1.4.2 The variational problem	22
1.4.3 The scalar case	23

1.4.4	Application to optimal design in the vector-valued case . . .	24
1.5	Some unsolved problems	26
1.5.1	Selection criterion	26
1.5.2	Measurable Hamiltonians	26
1.5.3	Lipschitz boundary data	27
1.5.4	Approximation of Lipschitz functions by smooth functions	27
1.5.5	Extension of Lipschitz functions and compatibility conditions	27
1.5.6	Existence under quasiconvexity assumption	28
1.5.7	Problems with constraints	28
1.5.8	Potential wells	29
1.5.9	Calculus of variations	30
I	First and Second Order PDE's	31
2	First Order Equations	33
2.1	Introduction	33
2.2	The convex case	34
2.2.1	The main theorem	34
2.2.2	An approximation lemma	36
2.2.3	The case independent of (x, u)	40
2.2.4	Proof of the main theorem	43
2.3	The nonconvex case	47
2.3.1	The pyramidal construction	47
2.3.2	The general case	52
2.4	The compatibility condition	56
2.5	An attainment result	60
3	Second Order Equations	69
3.1	Introduction	69
3.2	The convex case	70
3.2.1	Statement of the result and some examples	70
3.2.2	The approximation lemma	72
3.2.3	The case independent of lower order terms	73
3.2.4	Proof of the main theorem	77
3.3	Some extensions	81
3.3.1	Systems of convex functions	81
3.3.2	A problem with constraint on the determinant	82
3.3.3	Application to optimal design	90
4	Comparison with Viscosity Solutions	95
4.1	Introduction	95
4.2	Definition and examples	97
4.3	Geometric restrictions	100

4.3.1	Main results	100
4.3.2	Proof of the main results	103
4.4	Appendix	113
4.4.1	Subgradient and differentiability of convex functions	113
4.4.2	Gauges and their polars	113
4.4.3	Extension of Lipschitz functions	115
4.4.4	A property of the sub and super differentials	117
II Systems of Partial Differential Equations		119
5	Some Preliminary Results	121
5.1	Introduction	121
5.2	Different notions of convexity	121
5.2.1	Definitions and basic properties (first order case)	121
5.2.2	Definitions and basic properties (higher order case)	124
5.2.3	Different envelopes	126
5.3	Weak lower semicontinuity	127
5.3.1	The first order case	127
5.3.2	The higher order case	129
5.4	Different notions of convexity for sets	130
5.4.1	Definitions	130
5.4.2	The different convex hulls	131
5.4.3	Further properties of rank one convex hulls	135
5.4.4	Extreme points	138
6	Existence Theorems for Systems	141
6.1	Introduction	141
6.2	An abstract result	142
6.2.1	The relaxation property	142
6.2.2	Weakly extreme sets	147
6.3	The key approximation lemma	148
6.4	Sufficient conditions for the relaxation property	152
6.4.1	One quasiconvex equation	152
6.4.2	The approximation property	153
6.4.3	Relaxation property for general sets	154
6.5	The main theorems	157
III Applications		167
7	The Singular Values Case	169
7.1	Introduction	169
7.2	Singular values and functions of singular values	171
7.2.1	Singular values	171

7.2.2	Functions depending on singular values	174
7.2.3	Rank one convexity in dimension two.	181
7.3	Convex and rank one convex hulls	185
7.3.1	The case of equality of the a_i	186
7.3.2	The main theorem for general matrices	187
7.3.3	The diagonal case in dimension two	193
7.3.4	The symmetric case in dimension two	195
7.4	Existence of solutions (the first order case)	199
7.5	Existence of solutions (the second order case)	200
8	The Case of Potential Wells	205
8.1	Introduction	205
8.2	The rank one convex hull	206
8.3	Existence of solutions	215
9	The Complex Eikonal Equation	217
9.1	Introduction	217
9.2	The convex and rank one convex hulls	218
9.3	Existence of solutions	222
IV	Appendix	223
10	Appendix: Piecewise Approximations	225
10.1	Vitali covering theorems and applications	225
10.1.1	Vitali covering theorems	225
10.1.2	Piecewise affine approximation	232
10.2	Piecewise polynomial approximation	241
10.2.1	Approximation of functions of class C^N	242
10.2.2	Approximation of functions of class $W^{N,\infty}$	245
	References	249
	Index	271