

# CONTENTS

<b>1</b>	<b>Introduction and overview</b>	1
1.1	The planar theory	4
1.2	$n$ -Dimensional quasiconformal mappings	13
1.3	The Liouville theorem	16
1.4	Higher integrability	17
1.5	Stability and rigidity phenomena	18
1.6	Quasiconformal structures on manifolds	19
1.7	Nevanlinna theory	23
1.8	Non-linear potential theory	25
1.9	Singular integral operators	25
1.10	Removable singularities	27
1.11	Quasiconformal groups, semigroups and dynamics	27
1.12	Continuum mechanics and non-linear elasticity	29
1.13	Mostow rigidity	31
<b>2</b>	<b>Conformal mappings</b>	32
2.1	The Cauchy–Riemann system	32
2.2	The Möbius group	34
2.3	The Liouville theorem (smooth case)	36
2.4	Curvature	36
2.5	Computing the Jacobian	39
2.6	Conclusions	40
2.7	Further aspects	41
<b>3</b>	<b>Stability of the Möbius group</b>	43
3.1	Mapping classes	43
3.2	Harnack inequalities	45
3.3	A stability function	47
3.4	Passing Harnack inequalities on to $\mathcal{M}_t$	48
3.5	Local injectivity	50
<b>4</b>	<b>Sobolev theory and function spaces</b>	53
4.1	Schwartz distributions	53
4.2	Definitions of Sobolev spaces	57
4.3	Mollification	57
4.4	Lebesgue points	58
4.5	Pointwise coincidence of Sobolev functions	59

4.6	Alternative characterizations	60
4.7	Cross product of gradient fields	63
4.8	The adjoint differential	65
4.9	Subharmonic distributions	67
4.10	Embedding theorems	68
4.11	Duals and compact embeddings	73
4.12	Orlicz–Sobolev spaces	74
4.13	Hardy spaces and BMO	80
<b>5</b>	<b>The Liouville theorem</b>	85
5.1	Introduction	85
5.2	Second-order estimates	87
5.3	Identities	90
5.4	Second-order equations	93
5.5	Continuity of the Jacobian	95
5.6	A formula for the Jacobian	97
5.7	Concluding arguments	98
<b>6</b>	<b>Mappings of finite distortion</b>	99
6.1	Differentiability	100
6.2	Integrability of the Jacobian	104
6.3	Absolute continuity	105
6.4	Distortion functions	108
6.5	Examples	112
6.5.1	Radial stretchings	112
6.5.2	Winding maps	115
6.5.3	Cones and cylinders	118
6.5.4	The Zorich exponential map	119
6.5.5	A regularity example	122
6.5.6	Squeezing the Sierpinski sponge	126
6.5.7	Releasing the sponge	134
<b>7</b>	<b>Continuity</b>	138
7.1	Distributional Jacobians	140
7.2	The $L^1$ integrability of the Jacobian	143
7.3	Weakly monotone functions	148
7.4	Oscillation in a ball	150
7.5	Modulus of continuity	152
7.6	Exponentially integrable outer distortion	156
7.7	Hölder estimates	160
7.8	Fundamental $L^p$ -inequality for the Jacobian	163
7.8.1	A class of Orlicz functions	164
7.8.2	Another proof of Corollary 7.2.1	166

<b>8 Compactness</b>	169
8.1 Distributional Jacobians revisited	169
8.2 Weak convergence of Jacobians	172
8.3 Maximal inequalities	175
8.4 Improving the degree of integrability	176
8.5 Weak limits and orientation	181
8.6 $L \log L$ integrability	185
8.7 A limit theorem	186
8.8 Polyconvex functions	187
8.8.1 Null Lagrangians	188
8.8.2 Polyconvexity of distortion functions	190
8.9 Biting convergence	191
8.10 Lower semicontinuity of the distortion	193
8.11 The failure of lower semicontinuity	197
8.12 Bounded distortion	200
8.13 Local injectivity revisited	201
8.14 Compactness for exponentially integrable distortion	205
<b>9 Topics from Multilinear Algebra</b>	208
9.1 The $l$ -covectors	208
9.2 The wedge product	209
9.3 Orientation	211
9.4 The pullback	211
9.5 Matrix representations	212
9.6 Inner products	213
9.7 The volume element	216
9.8 Hodge duality	217
9.9 Hadamard–Schwarz inequality	220
9.10 Submultiplicity of the distortion	221
<b>10 Differential Forms</b>	222
10.1 Differential forms in $\mathbb{R}^n$	222
10.2 Pullback of differential forms	228
10.3 Integration by parts	229
10.4 Orlicz–Sobolev spaces of differential forms	232
10.5 The Hodge decomposition	234
10.6 The Hodge decomposition in $\mathbb{R}^n$	236
<b>11 Beltrami equations</b>	240
11.1 The Beltrami equation	240
11.2 A fundamental example	244
11.2.1 The construction	245
11.3 Liouville-type theorem	250

<b>11.4</b>	The principal solution	251
<b>11.5</b>	Stoilow factorization	253
<b>11.6</b>	Failure of factorization	255
<b>11.7</b>	Solutions for integrable distortion	257
<b>11.8</b>	Distortion in the exponential class	259
11.8.1	An example	261
11.8.2	Statement of results	262
<b>11.9</b>	Distortion in the subexponential class	264
11.9.1	An example	264
11.9.2	Statement of results	265
11.9.3	Further generalities	267
<b>11.10</b>	Preliminaries	268
11.10.1	Results from harmonic analysis	269
11.10.2	Existence for exponentially integrable distortion	270
11.10.3	Uniqueness	276
11.10.4	Critical exponents	278
11.10.5	Existence for subexponentially integrable distortion	280
<b>11.11</b>	Global solutions	284
<b>11.12</b>	Holomorphic dependence	289
<b>11.13</b>	Examples and non-uniqueness	292
<b>11.14</b>	Compactness	299
<b>11.15</b>	Removable singularities	300
<b>11.16</b>	Final comments	301
<b>12</b>	<b>Riesz transforms</b>	303
<b>12.1</b>	Singular integral operators	303
<b>12.2</b>	Fourier multipliers	308
<b>12.3</b>	Trivial extension of a scalar operator	312
<b>12.4</b>	Extension to $\mathbb{C}^n$	313
<b>12.5</b>	The real method of rotation	315
<b>12.6</b>	The complex method of rotation	316
<b>12.7</b>	Polarization	319
<b>12.8</b>	The tensor product of Riesz transforms	321
<b>12.9</b>	Dirac operators and the Hilbert transform on forms	323
<b>12.10</b>	The $L^p$ -norms of the Hilbert transform on forms	330
<b>12.11</b>	Further estimates	332
<b>12.12</b>	Interpolation	333
<b>13</b>	<b>Integral estimates</b>	337
<b>13.1</b>	Non-linear commutators	337
<b>13.2</b>	The complex method of interpolation	340

13.3	Jacobians and wedge products revisited	343
13.4	The $H^1$ -theory of wedge products	345
13.5	An $L \log L$ inequality	347
13.6	Estimates beyond the natural exponent	350
13.7	Proof of the fundamental inequality for Jacobians	352
<b>14</b>	<b>The Gehring lemma</b>	354
14.1	A covering lemma	356
14.2	Calderón–Zygmund decomposition	357
14.3	Gehring’s lemma in Orlicz spaces	359
14.4	Caccioppoli’s inequality	363
14.5	The order of zeros	367
<b>15</b>	<b>The governing equations</b>	370
15.1	Equations in the plane	370
15.2	Absolute minima of variational integrals	375
15.3	Conformal mappings	380
15.4	Equations at the level of exterior algebra	386
15.5	Even dimensions	391
15.6	Signature operators	393
15.7	Four dimensions	398
<b>16</b>	<b>Topological properties of mappings of bounded distortion</b>	401
16.1	The energy integrand	402
16.2	The Dirichlet problem	405
16.3	The $\mathcal{A}$ -harmonic equation	406
16.4	Caccioppoli inequality	410
16.5	The comparison principle	410
16.6	The polar set	411
16.7	Sets of zero conformal capacity	414
16.8	Qualitative analysis near polar points	416
16.9	Local injectivity of smooth mappings	419
16.10	The Jacobian is non-vanishing	422
16.11	Analytic degree theory	423
16.12	Openness and discreteness for mappings of bounded distortion	426
16.13	Further generalities	427
16.14	An update	428
<b>17</b>	<b>Painlevé’s theorem in space</b>	431
17.1	Painlevé’s theorem in the plane	431
17.2	Hausdorff dimension and capacity	432

17.3	Removability of singularities	434
17.4	Distortion of dimension	437
<b>18</b>	<b>Even dimensions</b>	440
18.1	The Beltrami operator	441
18.2	Integrability theorems in even dimensions	443
18.3	Mappings with exponentially integrable distortion	446
18.4	The $L^2$ inverse of $\mathbf{I} - \mu \mathcal{S}$	449
18.5	$W^{1,n}$ -regularity	452
18.6	Singularities	460
18.7	An example	461
<b>19</b>	<b>Picard and Montel theorems in space</b>	467
19.1	Picard's theorem in space	468
19.2	Serrin's theorem and Harnack functions	468
19.3	Estimates in $\mathcal{H}_\Theta(\mathbb{R}^n)$	469
19.4	Harnack inequalities near zeros	472
19.5	Collections of Harnack functions	475
19.6	Proof of Rickman's theorem	477
19.7	Normal families	480
19.8	Montel's theorem in space	483
19.9	Further generalizations	484
<b>20</b>	<b>Conformal structures</b>	486
20.1	The space $S(n)$	486
20.2	Conformal structures	489
20.3	The smallest ball	491
<b>21</b>	<b>Uniformly quasiregular mappings</b>	493
21.1	A first uniqueness result	494
21.2	First examples	496
21.3	Fatou and Julia sets	499
21.4	Lattès-type examples	501
21.5	Invariant conformal structures	505
<b>22</b>	<b>Quasiconformal groups</b>	510
22.1	Convergence properties	511
22.2	The elementary quasiconformal groups	513
22.3	Non-elementary quasiconformal groups	517
22.4	The triple space	519
22.5	Conjugacy results	520
22.6	Hilbert-Smith conjecture	524
22.7	Remarks	527

<b>23</b>	<b>Analytic continuation for Beltrami systems</b>	528
23.1	Uniqueness .	528
23.2	Proof of Theorem 23.1.1	529
23.3	Remarks	530
<b>Bibliography</b>		531
<b>Index</b>		547