CONTENTS

PREFACE iii

CHAPTER I. THEORY OF RIEMANN SURFACES: A QUICK REVIEW

§1. Topology of Riemann Surfaces 1
1. Exhaustion 1
2. The Homology Groups 2
3. The Fundamental Group 3
§2. Classical Potential Theory 4
4. Superharmonic Functions 4
5. The Dirichlet Problem 5
6. Potential Theory 7
§3. Differentials 9
7. Basic Definition 9
8. The Class Γ and its Subclasses 11
9. Cycles and Differentials 12
10. Riemann-Roch Theorem 14
11. Cauchy Kernels on Compact Bordered Surfaces 17
Notes 22

CHAPTER II. MULTIPLICATIVE ANALYTIC FUNCTIONS

§1. Multiplicative Analytic Functions 23
1. The First Cohomology Group 23
2. Line Bundles and Multiplicative Analytic Functions . . . 28
3. Existence of Holomorphic Sections 31
§2. Lattice Structure of Harmonic Functions 33
4. Basic Structure 33
5. Orthogonal Decomposition 36
Notes 38

CHAPTER III. MARTIN COMPACTIFICATION

§1. Compactification 39
1. Definition 39
2. Integral Representation 40
3. The Dirichlet Problem 43

§2. Fine Limits 49
4. Definition of Fine Limits 49
5. Analysis of Boundary Behavior 50
§3. Covering Maps 57
6. Correspondence of Harmonic Functions 57
7. Preservation of Harmonic Measures 59
Notes 63

CHAPTER IV. HARDY CLASSES

§1. Hardy Classes on the Unit Disk 64
1. Basic Definitions 64
2. Some Classical Results 66
§2. Hardy Classes on Hyperbolic Riemann Surfaces 73
3. Boundary Behavior of H^p and h^p Functions 73
4. Some Results on Multiplicative Analytic Functions . . . 74
5. The β-Topology 75
Notes 82

CHAPTER V. RIEMANN SURFACES OF PARREAU-WIDOM TYPE

§1. Definitions and Fundamental Properties 83
1. Basic Definitions 83
2. Widom's Characterization 85
3. Regularization of Surfaces of Parreau-Widom Type 86
§2. Proof of Widom's Theorem (I) 90
4. Analysis on Regular Subregions 90
5. Proof of Necessity 95
§3. Proof of Widom's Theorem (II) 99
6. Review of Principal Operators 99
7. Modified Green Functions 102
8. Proof of Sufficiency 111
9. A Few Direct Consequences 117
Notes 118

CHAPTER VI. GREEN LINES

§1. The Dirichlet Problem on the Space of Green Lines 119
1. Definition of Green Lines 119
2. The Dirichlet Problem 121
§2. The Space of Green Lines on a Surface of Parreau-Widom Type 124
3. The Green Star Regions 124
4. Limit along Green Lines 129

§3. The Green Lines and the Martin Boundary 132
5. Convergence of Green Lines 132
6. Green Lines and the Martin Boundary 135
7. Boundary Behavior of Analytic Maps 140
Notes 143

CHAPTER VII. CAUCHY THEOREMS

§1. The Inverse Cauchy Theorem 144
1. Statement of Results 144
2. Proof of Theorem 1B 145
§2. The Direct Cauchy Theorem 151
3. Formulation of the Condition 151
4. The Direct Cauchy Theorem of Weak Type 152
§3. Applications 155
5. Weak-star Maximality of H^∞ 155
6. Common Inner Factors 156
7. The Orthocomplement of $H^\infty(d\chi)$ 157
Notes 159

CHAPTER VIII. SHIFT-INVARIANT SUBSPACES

§1. Preliminary Observations 160
1. Generalities 160
2. Shift-Invariant Subspaces on the Unit Disk 162
§2. Invariant Subspaces 167
3. Doubly Invariant Subspaces 167
4. Simply Invariant Subspaces 169
5. Equivalence of (DCT_a) 177
Notes 178

CHAPTER IX. CHARACTERIZATION OF SURFACES OF PARREAU-WIDOM TYPE

§1. The Inverse Cauchy Theorem and Surfaces of Parreau-Widom Type 179
1. Statement of the Main Result 179
2. A Mean Value Theorem 183
3. Proof of the Main Theorem 187
§2. Conditions Equivalent to the Direct Cauchy Theorem 198
4. General Discussion 198
5. Functions $m^p(\xi,a)$ and (DCT) 200
Notes 207

CHAPTER X. EXAMPLES OF SURFACES OF PARREAU-WIDOM TYPE

§1. PWS of Infinite Genus for Which (DCT) Holds 208
1. PWS's of Myrberg Type 208
2. Verification of (DCT) 213
§2. Plane Regions of Parreau-Widom Type for Which (DCT) Fails . 215
3. Some Simple Lemmas 215
4. Existence Theorem 217
§3. Further Properties of PWS 221
5. Embedding into the Maximal Ideal Space 221
6. Density of $H^\infty(R)$ 223
§4. The Corona Problem for PWS 227
7. (DCT) and the Corona Theorem: Positive Examples 227
8. Negative Examples 229
Notes 233

CHAPTER XI. CLASSIFICATION OF PLANE REGIONS

§1. Hardy-Orlicz Classes 234
1. Definitions 234
2. Some Basic Properties 235
§2. Null Sets of Class N_ϕ 238
3. Preliminary Lemmas 238
4. Existence of Null Sets 247
§3. Classification of Plane Regions 253
5. Lemmas 253
6. Classification Theorem 256
7. Majoration by Quasibounded Harmonic Functions 260
Notes 261

APPENDICES

A.1. The Classical Fatou Theorem 262
A.2. Kolmogorov's Theorem on Conjugate Functions 267
A.3. The F. and M. Riesz Theorem 269

References 272

Index of Notations 276

Index 278