

## CONTENTS

Preface . . . . .	5
Acknowledgements . . . . .	7

### INTRODUCTION

§ 1. Iterates and orbits . . . . .	13
§ 2. Attractive and repulsive fixed points . . . . .	17
§ 3. A real variable . . . . .	19
§ 4. A complex variable . . . . .	24
§ 5. Functional equations . . . . .	25

### CHAPTER I

#### GENERAL SOLUTION

§ 1. Formulation of the problem and preliminaries . . . . .	29
§ 2. Construction of the general solution . . . . .	30
§ 3. Non-invertible $f$ . . . . .	35
§ 4. Automorphic functions . . . . .	41
§ 5. Abel's equation and Schröder's equation . . . . .	43
§ 6. General remarks . . . . .	44

### CHAPTER II

#### LINEAR EQUATION

§ 1. Solution depending on an arbitrary function . . . . .	46
§ 2. Homogeneous equation . . . . .	47
§ 3. Some criteria . . . . .	51
§ 4. Non-homogeneous equation . . . . .	52
§ 5. Case $ g(\xi)  > 1$ . . . . .	53
§ 6. Case $ g(\xi)  < 1$ . . . . .	54
§ 7. Case $g(x) \equiv \pm 1$ . . . . .	57
§ 8. Examples . . . . .	65

### CHAPTER III

#### CONTINUOUS SOLUTIONS

§ 1. Solution depending on an arbitrary function . . . . .	67
§ 2. Extensions of solutions . . . . .	70
§ 3. Unique solution . . . . .	72

§ 4. Lack of uniqueness . . . . .	75
§ 5. The function $f(x)$ decreasing . . . . .	77
§ 6. Examples . . . . .	81

## CHAPTER IV

## DIFFERENTIABLE SOLUTIONS

§ 1. Preliminaries . . . . .	84
§ 2. Solution depending on an arbitrary function . . . . .	87
§ 3. Existence theorem . . . . .	90
§ 4. Uniqueness theorem . . . . .	94
§ 5. Regularity of solutions . . . . .	98
§ 6. Lack of uniqueness . . . . .	100
§ 7. An application: the Goursat problem for a hyperbolic equation . . . . .	101
§ 8. Other examples . . . . .	103

## CHAPTER V

## MONOTONIC AND CONVEX SOLUTIONS

§ 1. Fundamental theorem . . . . .	106
§ 2. Monotonic solutions . . . . .	107
§ 3. Lack of uniqueness . . . . .	110
§ 4. Further uniqueness theorems . . . . .	113
§ 5. A finite difference equation . . . . .	114
§ 6. Generalization of the previous result . . . . .	118
§ 7. Recurrent sequences . . . . .	121
§ 8. General linear recurrence . . . . .	123
§ 9. Consequences for functional equations . . . . .	126
§10. Euler's Gamma function . . . . .	127
§11. Application to branching processes . . . . .	131

## CHAPTER VI

## SCHRÖDER'S EQUATION

§ 1. Preliminaries . . . . .	135
§ 2. Regular case . . . . .	137
§ 3. Koenigs existence theorem . . . . .	139
§ 4. Convex solutions . . . . .	141
§ 5. Principal solution . . . . .	143
§ 6. Singular case. Multiplier zero . . . . .	145
§ 7. Case $ s =1$ . A review of the results . . . . .	147
§ 8. Case of a root of unity . . . . .	148
§ 9. Divergence case . . . . .	149
§10. Convergence case . . . . .	149
§11. Conjugacy problem . . . . .	156
§12. Exponential and logarithmic functions . . . . .	159

CHAPTER VII  
ABEL'S EQUATION

§ 1. General . . . . .	163
§ 2. Asymptotic conditions at a finite fixed point . . . . .	167
§ 3. Convex solutions . . . . .	171
§ 4. A condition for the effectiveness of the Lévy algorithm . . . . .	173
§ 5. Exponentially growing functions . . . . .	174

CHAPTER VIII  
ANALYTIC SOLUTIONS

§ 1. Special homogeneous equation . . . . .	180
§ 2. Special inhomogeneous equation . . . . .	183
§ 3. Entire solutions of the homogeneous equation . . . . .	185
§ 4. General equation . . . . .	187
§ 5. The Gamma function in a complex variable . . . . .	191
§ 6. Riemann's Zeta function . . . . .	193

CHAPTER IX  
ITERATION

§ 1. Iteration groups . . . . .	197
§ 2. Regular iteration . . . . .	199
§ 3. Multiplier zero . . . . .	202
§ 4. Lévy iterates . . . . .	204
§ 5. Regular iteration at infinity . . . . .	206
§ 6. Analytic iteration . . . . .	209

CHAPTER X  
COMMUTING FUNCTIONS

§ 1. The real case . . . . .	213
§ 2. Semipermutable polynomials . . . . .	215
§ 3. Permutable entire functions . . . . .	218
§ 4. Exponential function . . . . .	222

CHAPTER XI  
SIMULTANEOUS EQUATIONS

§ 1. Biperiodic functions . . . . .	227
§ 2. Periodic solutions of functional equations . . . . .	230
§ 3. Further properties of the Gamma function . . . . .	232
§ 4. A continuous curve filling a square . . . . .	236
§ 5. Cantor's singular function . . . . .	241

## CHAPTER XII

## EQUATIONS OF HIGHER ORDERS AND SYSTEMS OF EQUATIONS

§ 1. Systems of equations . . . . .	244
§ 2. Equations and systems of equations of order $m$ . . . . .	246
§ 3. Uniqueness theorems . . . . .	248
§ 4. Lack of uniqueness . . . . .	252
§ 5. Gaussian normal distribution . . . . .	254

## CHAPTER XIII

## LINEAR EQUATIONS OF HIGHER ORDERS

§ 1. Reduction of order . . . . .	259
§ 2. Equation with constant coefficients . . . . .	262
§ 3. Finite groups of substitutions . . . . .	267
§ 4. Characterization of polynomials . . . . .	271

## CHAPTER XIV

## INVARIANT CURVES

§ 1. Unique invariant curve . . . . .	274
§ 2. Lack of uniqueness . . . . .	278
§ 3. A problem of continuation . . . . .	282
§ 4. Euler's equation . . . . .	286

## CHAPTER XV

## FRACTIONAL ITERATES

§ 1. The Babbage equation . . . . .	288
§ 2. Fractional iterates . . . . .	293
§ 3. Continuous increasing solutions . . . . .	297
§ 4. Continuous decreasing solutions for decreasing $g$ . . . . .	299
§ 5. Continuous decreasing solutions for increasing $g$ . . . . .	300
§ 6. Regular solutions . . . . .	303
§ 7. A generalization . . . . .	305
Bibliography . . . . .	308
Index of symbols . . . . .	373
Subject index . . . . .	375
Index of names . . . . .	377