Table of Contents

Foreword to the second Russian edition vii

CHAPTER 1. General Theorems on Elliptic Functions 1
§1. On periods of single-valued analytic functions 1
§2. Proof of the Jacobi theorem 2
§3. Theta functions 4
§4. Liouville's theorems 6
§5. The Weierstrass function $\wp(u)$ 9
§6. The differential equation of the function $\wp(u)$ 12

CHAPTER 2. Modular Functions 15
§7. Invariants 15
§8. Modular forms 18
§9. Fundamental regions of the group Σ 22
§10. The modular function $J(\tau)$ 27
§11. Inversion of elliptic integrals of the first kind 32

CHAPTER 3. Weierstrass Functions 35
§12. The Weierstrass function $\zeta(u)$ 35
§13. The Weierstrass function $\sigma(u)$ 37
§14. Expression of an arbitrary elliptic function by means of $\sigma(u)$ and by means of $\zeta(u)$ 38
§15. The addition theorems for Weierstrass functions 40
§16. Representation of every elliptic function in terms of the functions $\wp(u)$ and $\wp'(u)$ 43
§17. Elliptic integrals 45

CHAPTER 4. Theta Functions 49
§18. Representations of theta functions by infinite products 49
§19. The connection between sigma functions and theta functions 52
§20. Expansion of the functions $\zeta(u)$ and $\wp(u)$ in simple series 54

§21. Expressions for e_1, e_2, and e_3 in terms of the zero values of the theta functions 55
§22. Transformation of theta functions 56
§23. The modular function $\lambda(\tau)$ 58

CHAPTER 5. Jacobi Functions 65
§24. The elliptic integral of the first kind in the forms of Jacobi and Riemann 65
§25. The Jacobi functions 67
§26. Differentiation of the Jacobi functions 70
§27. The Jacobi function $Z(w)$ 71
§28. The Euler theorem 72
§29. Normal elliptic integrals of the second and third kinds in Jacobi form 75
§30. Complete elliptic integrals of the first kind 77
§31. Complete elliptic integrals of the second kind 80
§32. Degeneracy of elliptic functions 82
§33. The simple pendulum 84

CHAPTER 6. Transformation of Elliptic Functions 87
§34. The problem of transformation of elliptic functions 87
§35. Reduction of the general problem 89
§36. The first principal first-degree transformation 93
§37. The second principal first-degree transformation 94
§38. Landen's transformation 95
§39. Gauss's transformation 96
§40. Principal transformations of nth degree 98

CHAPTER 7. Additional Facts About Elliptic Integrals 101
§41. Elliptic curves of general form 101
§42. The function $\wp(u)$ with real invariants 104
§43. Reduction of elliptic integrals to Jacobi normal form in the real case 106
§44. Complete elliptic integrals as hypergeometric functions 109
§45. Computation of h from a given modulus k 114
§46. The arithmetico-geometric mean 115

CHAPTER 8. Some Conformal Mappings 119
§47. A conformal mapping of a rectangle onto a half-plane 119
§48. A conformal mapping of a doubly connected polygonal domain onto a circular annulus 127
§49. Examples of conformal mappings 133

CHAPTER 9. Extremal Properties of Fractions to Which a Transformation of Elliptic Functions Reduces 141
§50. Statement of the problems 141
§51. Solution of Problem C 147

CHAPTER 10. Generalization of Tchebycheff Polynomials 151
§52. Polynomials deviating least from zero 151
§53. Orthogonal polynomials on two intervals 156

CHAPTER 11. Various Supplements and Applications 163
§54. Abel's theorem 163
§55. The Green's function for a circular annulus 169
§56. The Dirichlet problem for a circular annulus 171
§57. Elliptic coordinates 175
§58. The Laplace equation in elliptic coordinates 181
§59. The Lamé equation 182
§60. The Picard theorems on meromorphic functions 187
§61. The Landau theorem 188
§62. On meromorphic functions having an algebraic addition theorem 190
§63. On Fourier series of analytic functions 191

Tables of the Most Important Formulas 197

Tables of Values of Elliptic Integrals 217

Bibliography 233

Subject Index 235