

Contents

Chapter One

Fundamentals of General Topology

§ 1. Topological spaces	1
1. The notion of a topological space	1
2. Neighbourhoods	2
3. Bases of neighbourhoods	3
4. Hausdorff spaces	3
5. Some simple topological ideas	4
6. Induced topologies and comparison of topologies. Connectedness	4
7. Continuous mappings	6
8. Topological products	7
§ 2. Nets and filters	9
1. Partially ordered and directed sets	9
2. ZORN's lemma	9
3. Nets in topological spaces	10
4. Filters	11
5. Filters in topological spaces	12
6. Nets and filters in topological products	13
7. Ultrafilters	14
8. Regular spaces	15
§ 3. Compact spaces and sets	16
1. Definition of compact spaces and sets	16
2. Properties of compact sets	17
3. TYCHONOFF's theorem	18
4. Other concepts of compactness	18
5. Axioms of countability	19
6. Locally compact spaces	20
7. Normal spaces	22
§ 4. Metric spaces	23
1. Definition	23
2. Metric space as a topological space	23
3. Continuity in metric spaces	24
4. Completion of a metric space	25
5. Separable and compact metric spaces	26
6. BAIRE's theorem	27
7. The topological product of metric spaces	28

§ 5. Uniform spaces	29
1. Definition	29
2. The topology of a uniform space	30
3. Uniform continuity	31
4. Cauchy filters	32
5. The completion of a Hausdorff uniform space	33
6. Compact uniform spaces	35
7. The product of uniform spaces	37
§ 6. Real functions on topological spaces	38
1. Upper and lower limits	38
2. Semi-continuous functions	40
3. The least upper bound of a collection of functions	41
4. Continuous functions on normal spaces	42
5. The extension of continuous functions on normal spaces	44
6. Completely regular spaces	44
7. Metrizable uniform spaces	45
8. The complete regularity of uniform spaces	47

Chapter Two

Vector Spaces over General Fields

§ 7. Vector spaces	48
1. Definition of a vector space	48
2. Linear subspaces and quotient spaces	50
3. Bases and complements	50
4. The dimension of a linear space	52
5. Isomorphism, canonical form	53
6. Sums and intersections of subspaces	54
7. Dimension and co-dimension of subspaces	55
8. Products and direct sums of vector spaces	56
9. Lattices	57
10. The lattice of linear subspaces	58
§ 8. Linear mappings and matrices	59
1. Definition and rules of calculation	59
2. The four characteristic spaces of a linear mapping	60
3. Projections	60
4. Inverse mappings	61
5. Representation by matrices	63
6. Rings of matrices	65
7. Change of basis	66
8. Canonical representation of a linear mapping	66
9. Equivalence of mappings and matrices	67
10. The theory of equivalence	68
§ 9. The algebraic dual space. Tensor products	69
1. The dual space	69
2. Orthogonality	70
3. The lattice of orthogonally closed subspaces of E^*	72

4. The adjoint mapping	73
5. The dimension of E^*	74
6. The tensor product of vector spaces	76
7. Linear mappings of tensor products	78
§ 10. Linearly topologized spaces	82
1. Preliminary remarks	82
2. Linearly topologized spaces	82
3. Dual pairs, weak topologies	85
4. The dual space	86
5. The dual pairs $\langle E^*, E \rangle$	88
6. Weak convergence and weak completeness	89
7. Quotient spaces and topological complements	90
8. Dual spaces of subspaces and quotient spaces	93
9. Linearly compact spaces	95
10. E^* as a linearly compact space	97
11. The topology \mathfrak{T}_{jk}	97
12. \mathfrak{T}_{jk} -continuous linear mappings	98
13. Continuous basis and continuous dimension	100
§ 11. The theory of equations in E and E^*	101
1. The duality of E and E^*	101
2. The theory of the solutions of column-and row-finite systems of equations	103
3. Formulae for solutions	104
4. The countable case	106
5. An example	107
§ 12. Locally linearly compact spaces	108
1. The structure of locally linearly compact spaces	108
2. The endomorphisms of ψ	109
3. The theory of equivalence in ψ	111
§ 13. The linear strong topology	113
1. Linearly bounded subspaces	113
2. The linear strong topology	114
3. The completion	115
4. Topological sums and products	117
5. Spaces of countable degree	119
6. A counterexample	120
7. Further investigations	121

Chapter Three

Topological Vector Spaces

§ 14. Normed spaces	123
1. Definition of a normed space	123
2. Norm isomorphism, equivalent norms	125
3. Banach spaces	126
4. Quotient spaces and topological products	127

5. The dual space	128
6. Continuous linear mappings	129
7. The spaces c_0 , c , l^1 and l^∞	130
8. The spaces l^p , $1 < p < \infty$	134
9. (B)-spaces of continuous and holomorphic functions	137
10. The L^p spaces ($p \geq 1$)	139
11. The space L^∞	142
§ 15. Topological vector spaces	144
1. Definition of a topological vector space	144
2. A second definition	146
3. The completion	148
4. Quotient spaces and topological products	149
5. Finite dimensional topological vector spaces	151
6. Bounded and compact subsets	152
7. Locally compact topological vector spaces	155
8. Topologically complementary spaces	155
9. The dual space, hyperplanes, the spaces L^p with $0 < p < 1$	156
10. Locally bounded spaces, quasi-norms, p -norms	159
11. Metrizable spaces	162
12. The BANACH-SCHAUDER theorem and the closed-graph theorem	166
13. Equicontinuous mappings, and the theorems of BANACH and BANACH-STEINHAUS	168
14. Bilinear mappings	171
§ 16. Convex sets	173
1. The convex and absolutely convex cover of a set	173
2. The algebraic boundary of a convex set	176
3. Half-spaces	179
4. Convex bodies and the Minkowski functionals associated with them	180
5. Convex cones	183
6. Hypercones	184
§ 17. The separation of convex sets. The HAHN-BANACH theorem	186
1. The separation theorem	186
2. The HAHN-BANACH theorem	188
3. The analytic proof of the HAHN-BANACH theorem	189
4. Two consequences of the HAHN-BANACH theorem	192
5. Supporting hyperplanes	193
6. The HAHN-BANACH theorem for normed spaces. Adjoint mappings	196
7. The dual space of $C(I)$	197

Chapter Four

Locally Convex Spaces. Fundamentals

§ 18. The definition and simplest properties of locally convex spaces	202
1. Definition by neighbourhoods, and by semi-norms	202
2. Metrizable locally convex spaces and (F)-spaces	204
3. Subspaces, quotient spaces and topological products of locally convex spaces	206

4. The completion of a locally convex space	208
5. The locally convex direct sum of locally convex spaces	211
§ 19. Locally convex hulls and kernels, inductive and projective limits of locally convex spaces	215
1. The locally convex hull of locally convex spaces	215
2. The inductive limit of vector spaces	217
3. The topological inductive limit of locally convex spaces	220
4. Strict inductive limits	222
5. (LB)- and (LF)-spaces. Completeness	223
6. The locally convex kernel of locally convex spaces	225
7. The projective limit of vector spaces	228
8. The topological projective limit of locally convex spaces	230
9. The representation of a locally convex space as a projective limit	231
10. A criterion for completeness	232
§ 20. Duality	233
1. The existence of continuous linear functionals	233
2. Dual pairs and weak topologies	234
3. The duality of closed subspaces	236
4. Duality of mappings	237
5. Duality of complementary spaces	238
6. The convex cover of a compact set	240
7. The separation theorem for compact convex sets	243
8. Polarity	245
9. The polar of a neighbourhood of \circ	247
10. A representation of locally convex spaces	249
11. Bounded and strongly bounded sets in dual pairs	251
§ 21. The different topologies on a locally convex space	254
1. The topology \mathfrak{I}_{un} of uniform convergence on \mathfrak{M}	254
2. The strong topology	256
3. The original topology of a locally convex space; separability	258
4. The Mackey topology	260
5. The topology of a metrizable space	262
6. The topology \mathfrak{I}_c of precompact convergence	263
7. Polar topologies	266
8. The topologies \mathfrak{I}^f and \mathfrak{I}^{f^*}	267
9. GROTHENDIECK'S construction of the completion	269
10. The BANACH-DIEDONNÉ theorem	272
11. Real and complex locally convex spaces	273
§ 22. The determination of various dual spaces and their topologies	275
1. The dual of subspaces and quotient spaces	275
2. The topologies of subspaces, quotient spaces and their duals	276
3. Subspaces and quotient spaces of normed spaces	279
4. The quotient spaces of l^1	280
5. The duality of topological products and locally convex direct sums	283
6. The duality of locally convex hulls and kernels	288
7. Topologies on locally convex hulls and kernels	291

Chapter Five

Topological and Geometrical Properties of Locally Convex Spaces

§ 23. The bidual space. Semi-reflexivity and reflexivity	295
1. Quasi-completeness	295
2. The bidual space	297
3. Semi-reflexivity	298
4. The topologies on the bidual	300
5. Reflexivity	302
6. The relationship between semi-reflexivity and reflexivity	304
7. Distinguished spaces	306
8. The dual of a semi-reflexive space	307
9. Polar reflexivity	308
§ 24. Some results on compact and on convex sets	310
1. The theorems of ŠMULIAN and KAPLANSKY	310
2. EBERLEIN's theorem	313
3. Further criteria for weak compactness	315
4. Convex sets in spaces which are not semi-reflexive. The theorems of KLEE	319
5. KREIN's theorem	323
6. PTÁK's theorem	326
§ 25. Extreme points and extreme rays of convex sets	330
1. The KREIN-MILMAN theorem	330
2. Examples and applications	333
3. Variants of the KREIN-MILMAN theorem	336
4. The extreme rays of a cone	337
5. Locally compact convex sets	339
§ 26. Metric properties of normed spaces	342
1. Strict convexity	342
2. Shortest distance	343
3. Points of smoothness	345
4. Weak differentiability of the norm	347
5. Examples	350
6. Uniform convexity	353
7. The uniform convexity of the l^p and L^p spaces	355
8. Further examples	359
9. Invariance under topological isomorphisms	360
10. Uniform smoothness and strong differentiability of the norm	363
11. Further ideas	366

Chapter Six

Some Special Classes of Locally Convex Spaces

§ 27. Barrelled spaces and Montel spaces	367
1. Quasi-barrelled spaces and barrelled spaces	367
2. (M)-spaces and (FM)-spaces	369

3. The space $H(\mathfrak{G})$	372
4. (M)-spaces of locally holomorphic functions	375
§ 28. Bornological spaces	379
1. Definition	379
2. The structure of bornological spaces	380
3. Local convergence. Sequentially continuous mappings	382
4. Hereditary properties	383
5. The dual, and the topology \mathfrak{T}_{c_0}	384
6. Boundedly closed spaces	386
7. Reflexivity and completeness	388
8. The MACKEY-ULAM theorem	389
§ 29. (F)- and (DF)-spaces	392
1. Fundamental sequences of bounded sets. Metrizable	392
2. The bidual	394
3. (DF)-spaces	396
4. Bornological (DF)-spaces	399
5. Hereditary properties of (DF)-spaces	401
6. Further results, and open questions	403
§ 30. Perfect spaces	405
1. The α -dual. Examples	405
2. The normal topology of a sequence space	407
3. Sums and products of sequence spaces	409
4. Unions and intersections of sequence spaces	410
5. Topological properties of sequence spaces	412
6. Compact subsets of a perfect space	415
7. Barrelled spaces and (M)-spaces	417
8. Echelon and co-echelon spaces	419
9. Co-echelon spaces of type (M)	421
10. Further investigations into sequence spaces	423
§ 31. Counterexamples	424
1. The dual of l^∞	424
2. Subspaces of l^∞ and l^1 with no topological complements	426
3. The problem of complements in l^p and L^p	428
4. Complements in (F)-spaces	431
5. An (FM)-space	433
6. An (LB)-space which is not complete	434
7. An (F)-space which is not distinguished	435
Bibliography	437
Author and Subject Index	447