TABLE OF CONTENTS

CHAPTER 0 PRELIMINARIES 1

0.1. Notations and Terminology 1

0.2. Vector Spaces 2

0.3. Sequences 3

0.4. Some Results of Integration 4

0.4.1. Set of measure zero on the line $\mathbb{R}$ 5

0.4.2. The saw-tooth function 5

CHAPTER 1 FINITE PARTS OF INTEGRALS 7

1.1. Definition 7

1.2. Extensions of the Definition 9

1.3. Integration by Parts 10

1.4. Analytic Continuation 12

1.5. Representations of Finite Parts on the Real Axis by Analytic Functions in the Complex Plane 15

1.6. Change of Variable 17

CHAPTER 2 BASE SPACES 19

2.1. Base Spaces 19

2.2. The Space $\mathcal{D}$ 19

2.3. The Space $\mathcal{D}^k$ $(k \geq 0)$ 20

2.4. The Space $\mathcal{S}$ (Functions of Rapid Descent) 20

2.5. The Space $\mathcal{E}$ 21

2.6. The Space $\mathcal{Z}$ (of Entire Functions) 21

2.7. Inclusions 21

2.8. The Space Φ 22

2.9. The Space $\Phi(\mathbb{R}^n)$ 23

CHAPTER 3 DEFINITION OF DISTRIBUTIONS 25

3.1. Generalized Functions 25

3.1.1. Inclusion of Φ' 26

3.2. Distributions 27

3.2.1. Inclusions 27
3.3. Examples of Distributions 27
3.3.1. Regular distributions 27
3.3.2. Irregular distributions 28
3.3.3. Pseudo functions 29
3.3.4. Regular tempered distributions 30
3.3.5. Tempered pseudo functions 30
3.3.6. Analytic functionals (ultradistributions) 31

CHAPTER 4 PROPERTIES OF GENERALIZED FUNCTIONS AND DISTRIBUTIONS 35

4.1. Support 35
4.1.1. Point support 36
4.1.2. Distributions with lower bounded support 36
4.1.3. Distributions with bounded support 37
4.2. Properties 37
4.2.1. Boundedness 37
4.3. Convergence 38
4.3.1. Completeness and limit 39
4.3.2. Particular cases of convergence in $\mathcal{D}'$ 39
4.3.3. Convergence in $\mathcal{S}'$ 40
4.3.4. Convergence to $\delta(x)$ 40
4.4. Approximation of Distributions by Regular Functions 41
4.5. Distributions in Several Variables 42

CHAPTER 5 OPERATIONS ON GENERALIZED FUNCTIONS AND DISTRIBUTIONS 47

5.1. Transpose of an Operation 47
5.2. Translation 48
5.3. Product by a Function 49
5.3.1. The space $M(\phi)$ and the general definition of product 50
5.3.2. Distributions belonging to $\mathcal{D}'$ or $\mathcal{E}'$ 50
5.3.2.1. Distributions of finite order 51
5.3.3. Tempered distributions 51
5.3.4. Ultradistribution 51
5.4. Differentiation 52
5.4.1. General outline 52
5.4.2. Remark 52
5.4.3. Distributions of finite order having bounded support 53

5.4.4. Derivatives of the Dirac distribution 53
5.4.5. Derivatives of a regular distribution 54
5.4.6. Derivatives of pseudo functions 57
5.4.7. Derivatives of ultradistributions 59
5.5. Differentiation of Product 59
5.6. Differentiation of Limit and Series 61
5.7. Derivatives in the Case of Several Variables 62
5.7.1. Generalization of $\delta'(x)$ 63
5.7.2. The Laplacian 64
5.8. Convolution 65
5.8.1. General definition 65
5.8.2. Convolution in $\mathbb{D}'$ 65
5.8.3. Examples 66
5.8.4. Convolution in $\mathbb{D}'_+$ 68
5.8.5. Convolution in $\mathcal{S}'$ 70
5.8.5.1. Convolution in $\mathcal{S}'_+$ 71
5.8.6. Convolution equations 71
5.8.7. Fundamental solution 71
5.9. Transformation of the Variable 72
5.9.1. Definition of $T_{u(x)}$ 72
5.9.2. Examples 74
5.9.3. Bibliography 75

CHAPTER 6 OTHER OPERATIONS ON DISTRIBUTIONS 77

6.1. Division 77
6.1.1. Division by x^n ($n>0$, an integer) 77
6.1.2. Division by a function 78
6.1.3. $\frac{1}{\alpha(x)}$ multiplier for Φ 79
6.2. Antidifferentiation 80
6.2.1. Antiderivative in $\mathbb{D}'_+$ 81

6.3. Value and Limit at a Point of a Distribution 82
6.3.1. Value at a point 82
6.3.2. Right and left hand limits at a point 83
6.3.3. Limit at infinity 84
6.4. Equivalence 85
6.4.1. Equivalence at the origin 85
6.4.2. Equivalence at infinity 88

CHAPTER 7 THE FOURIER TRANSFORMATION 91

7.1. Fourier Transformation on $\mathbb{Z}$ 91
7.2. Fourier Transformation on $\mathbb{D}$ 93

7.3. Fourier Transformation on $\mathcal{D}'$ and $\mathcal{Z}'$ 94
7.4. Inversion and Convergence 94
7.4.1. Inversion of Fourier transformation on $\mathcal{D}'$ and $\mathcal{Z}'$ 94
7.4.2. Convergence 95
7.5. Rules 96
7.6. Fourier Transformation on $\mathcal{E}'$ 96
7.7. Examples 97
7.8. Fourier Transformation on $\mathcal{S}$ and $\mathcal{S}'$ 99
7.9. Particular Cases 100
7.10. Examples 101
7.11. The Spaces $C(\mathcal{S}')$ and $M(\mathcal{S})$ of Fourier Transformation 101
7.12. The Fourier Transformation of Convolution and Multiplication 102
7.13. Applications 103
7.14. Bibliography 105

CHAPTER 8 THE LAPLACE TRANSFORMATION 107

8.1. Laplace Transformability 108
8.2. Laplace Transform 109
8.2.1. Case for functions 110
8.3. Characterization of Laplace Transform 110
8.4. Relation with the Fourier Transformation 113
8.5. Principal Rules 113
8.5.1. Case for functions 115
8.6. Convergence and Series 116
8.6.1. Examples 117
8.7. Inversion of the Laplace Transformation 118
8.7.1. Example 120
8.8. Reciprocity of the Convergence 120
8.8.1. Corollary in series 121
8.8.2. Examples 121
8.9. Differentiation with Respect to a Parameter 122
8.10. Laplace Transformation of Pseudo Functions 124
8.10.1. Derivative and primitive 124
8.10.2. Use of analytic continuation 125
8.10.3. Change of x to ax, a being complex 127
8.10.4. Change of x to ix 129
8.10.5. Convergence 131
8.11. Abelian Theorems 132
8.11.1. Behaviour of the transform at infinity 132

8.11.2. Behaviour of the transform near a singular point 134
8.12. Tauberian Theorems 136
8.12.1. Behaviour near the lower bound of the support 136
8.13. The n-Dimensional Laplace Transformation 138
8.13.1. The Laplace transformation in n variables 139
8.13.2. Convolution 140
8.14. Bibliography 143

CHAPTER 9 APPLICATIONS OF THE LAPLACE TRANSFORMATION 145

9.1. Convolution Equations 145
9.1.1. Examples 146
9.2. Differential Equations with Constant Coefficients 148
9.2.1. Solving distribution-derivative equations 148
9.2.2. Solving traditional differential equations 151
9.2.3. Single differential equations (Cauchy problems) 152
9.2.4. Systems of differential equations 154
9.3. Differential Equations with Polynomial Coefficients 155
9.3.1. Reduction of order 156
9.4. Integral Equations 160
9.4.1. Special Volterra equations 161
9.4.2. Resolvent series 162
9.4.3. Remark on uniqueness 164
9.4.4. Integral equations with polynomial coefficients 164
9.5. Integro-Differential Equations 166
9.6. General Concept of Green's Functions 168
9.6.1. Statement 168
9.6.2. Green's kernel 169
9.6.3. Examples 173
9.6.4. Integral equations 176
9.7. Partial Differential Equations 177
9.7.1. Diffusion of heat flow in rods 177
9.7.1.1. Infinite conductor without radiation 177
9.7.1.2. The cooling of a rod of finite length 179

9.7.1.3. Rod heated at an extremity 180
9.7.2. Vibrating strings 182
9.7.3. The telegraph equation 187
9.7.3.1. The lines without leakage which are closed by a resistance 187
9.7.3.2. The infinite line which is perfectly isolated 189
9.8. Convolution Formulae 190
9.9. Expansion in Series 193
9.9.1. Function $B(\nu,z)$ 193
9.9.2. Function $\psi(z)$ 194
9.9.3. Fourier series 194
9.9.4. Asymptotic expansions 196
9.10. Derivatives and Anti-Derivatives of Complex Order 198
9.10.1. Definition by the Laplace transformation 198
9.10.2. Examples 200
9.10.3. Extension of the definition 203
CHAPTER 10 THE STIELTJES TRANSFORMATION 207
10.1. The Spaces $E(r)$ and $\mathcal{I}'(r)$ 207
10.1.1. The space $E(r)$ 207
10.1.2. The space $\mathcal{I}'(r)$ 209
10.2. The Stieltjes Transformation 209
10.3. Iteration of the Laplace Transformation 210
10.4. Characterization of Stieltjes Transforms 211
10.5. Examples of Stieltjes Transforms 213
10.5.1. Examples when $T_t \in \mathcal{I}'(r)$ 213
10.5.2. Examples when $T_t \in \mathcal{I}'$ 215
10.6. Inversion 216
10.7. Abelian Theorems 219
10.7.1. Behaviour of the transform near the origin 219
10.7.2. Behaviour of the transform at infinity 220
10.8. The n-Dimensional Stieltjes Transformation 221
10.8.1. The space $J'_n(r)$ 221
10.8.2. The Stieltjes transformation in n variables 222
10.8.3. The iteration of the Laplace transformation 222
10.8.4. Inversion 224

10.9. Applications 224
10.10. Bibliography 224

CHAPTER 11 THE MELLIN TRANSFORMATION 227

11.1. Mellin Transformation of Functions 228
11.2. The Spaces $E_{\alpha,\omega}$ 230
11.3. The Spaces $E'_{\alpha,\omega}$ 232
11.3.1. The multiplication in $E'_{\alpha,\omega}$ 234
11.3.2. The differentiation in $E'_{\alpha,\omega}$ 234
11.3.3. Comparison with Zemanian spaces 235
11.4. The Mellin Transformation 236
11.5. Examples of Mellin Transforms 237
11.6. Characterization of Mellin Transformation 238
11.7. Rules of Calculus 241
11.8. Mellin and Laplace Transformations 242
11.9. Mellin and Fourier Transformations 244
11.10. Inversion of the Mellin Transformation 245
11.11. The Mellin Convolution 249
11.11.1. Examples and particular cases 250
11.11.2. Relation with the Mellin transformation 251
11.11.3. Relation with the ordinary convolution 251
11.11.4. The operator $(tD)^{\nu}$ 252
11.12. Abelian Theorems 253
11.13. Solution of Some Integral Equations 258
11.14. Euler-Cauchy Differential Equations 261
11.15. Potential Problems in Wedge Shaped Regions 265
11.16. Bibliography 268

CHAPTER 12 HANKEL TRANSFORMATION AND BESSEL SERIES 269

12.1. Hankel Transformation of Functions 269
12.2. The Spaces H_{ν} and H'_{ν} 272
12.3. Operations on H_{ν} and H'_{ν} 274
12.4. Hankel Transformation of Distributions 276
12.4.1. The Hankel transformation on $\mathcal{E}'(I)$ 280
12.5. Some Rules 282
12.5.1. Transform formulae for H_{ν} 282
12.5.2. Transform formulae for H'_{ν} 283
12.6. Inversion 284
12.6.1. Remarks 286
12.7. The n-Dimensional Hankel Transformation 287

12.7.1. The spaces of h_μ and h'_μ 288
12.7.2. Operations on h_μ and h'_μ 290
12.7.3. The Hankel transformation in n-variables 291
12.8. Variable Flow of Heat in Circular Cylinder 295
12.9. Bessel Series for Generalized Functions 297
12.9.1. Statement 297
12.10. The Space $B_{m,\nu}$ 298
12.11. Representation of a Distribution by its Fourier Bessel Series 300
12.12. Other Properties of the Fourier-Bessel Series 302
12.13. The Subspace B_m of $B_{m,\nu}$ 304
12.14. Bessel-Dini Series 307
12.14.1. Statement 307
12.15. The Space $B_{H,m,\nu}$ 309
12.16. Representation of a Distribution by its Bessel-Dini Series 310
12.16.1. The subspace B_m of $B_{H,m,\nu}$ 311
12.16.2. Another subspace of $B_{H,m,\nu}$ 311
12.17. An Application of the Bessel-Dini Series 311
12.18. Bibliography 314

BIBLIOGRAPHY 315

INDEX OF SYMBOLS 329

AUTHOR INDEX 331