Contents

Preface ix
Acknowledgments xiii
Some remarks concerning notation xv

Chapter I Some results on functional analysis

1 Basic concepts of order theory 1
1.1 Ordered spaces 2
1.2 Ordered linear spaces 2
1.3 Positive linear functionals and operators 9
1.4 Convergence and continuity 11
2 Convex sets 14
2.1 Basic concepts 14
2.2 Description by functionals 17
2.3 The special case of cones 19
3 Fixed-point theorems and other means of functional analysis 22
3.1 Compact operators in a Banach space 22
3.2 Iterative procedures in ordered spaces 24
Notes 26

Chapter II Inverse-positive linear operators

1 An abstract theory 29
1.1 The concept of an inverse-positive operator, general assumptions 30
1.2 Basic results 31
1.3 A more detailed analysis 33
1.4 Connected sets of operators 34
1.5 Noninvertible operators 36
2 M-matrices 37
2.1 Definitions and notation 38
2.2 General results for square matrices 39
2.3 Off-diagonally negative matrices 42
2.4 Eigenvalue problems 47
2.5 Iterative procedures 54
3 Differential operators of the second order, pointwise inequalities 61
3.1 Assumptions and notation 62
3.2 Basic results on inverse-positivity 64
3.3 Construction of majorizing functions 68
3.4 Sets of inverse-positive regular differential operators 71
3.5 Inverse-positivity under constraints 76
3.6 Stronger results on positivity in the interior 78
3.7 Strictly inverse-positive operators 81
3.8 Equivalence statements 85
3.9 Oscillation and separation of zeros 86
3.10 Eigenvalue problems 88
3.11 Differential operators on noncompact intervals 90
4 Abstract theory (continuation) 94
4.1 The method of reduction 95
4.2 Z-operators and M-operators 104
4.3 Operators $\lambda I - T$ with positive T 112
4.4 Inverse-positivity for the case $K^c = \emptyset$ 118
5 Numerical applications 121
5.1 A convergence theory for difference methods 122
5.2 A posteriori error estimates 132
6 Further results 144
6.1 Differential operators of the fourth order 144
6.2 Duality 148
6.3 A matrix of Lyapunov 149
Notes 150

Chapter III Two-sided bounds for second order differential equations

1 The concept of an inverse-monotone operator 160
2 Inverse-monotone differential operators 162
2.1 Simple sufficient conditions 162
2.2 More general results 166
2.3 Miscellaneous results 174
3 Inverse-monotonicity with weak differential inequalities 180
3.1 Description of the problem 181
3.2 Monotone definite operators 185

3.3 Further constructive sufficient conditions 188
3.4 L_1-differential inequalities 193
3.5 Stronger positivity statements for linear problems 195
4 Existence and inclusion for two-point boundary value problems 198
4.1 The basic theorem 200
4.2 Bounds for the derivative of the solution 203
4.3 Quadratic growth restrictions, Nagumo conditions 207
4.4 Singular problems 212
4.5 Different approaches 216
5 Existence and inclusion by weak comparison functions 220
5.1 Application of Schauder's fixed-point theorem 220
5.2 Existence by the theory on monotone definite operators 222
6 Error estimates 223
6.1 Construction of error bounds by linear approximation 223
6.2 Initial value problems 224
6.3 Boundary value problems 226
Notes 228

Chapter IV An estimation theory for linear and nonlinear operators, range-domain implications

1 Sufficient conditions, general theory 234
1.1 Notation and introductory remarks 234
1.2 A continuity principle 235
2 Inverse-monotone operators 237
2.1 Abstract results using majorizing elements 238
2.2 Nonlinear functions in finite-dimensional spaces 245
2.3 Functional-differential operators of the second order 248
2.4 A different approach for initial value problems 251
3 Inclusion by upper and lower bounds 259
4 Range-domain implications for linear and concave operators 261
4.1 Abstract theory 262
4.2 Applications to non-inverse-positive differential operators 264
Notes 272

Chapter V Estimation and existence theory for vector-valued differential operators

1 Description of the problem, assumptions, notation 275
2 Two-sided bounds by range-domain implications 279
3 Existence and inclusion by two-sided bounds 287
3.1 The basic theorem 288
3.2 Simultaneous estimation of the solution and its derivative 290
3.3 Growth restrictions concerning u' 293
3.4 Application of the methods to other problems 295
4 Pointwise norm bounds by range-domain implications 299
4.1 Notation and auxiliary means 299
4.2 General estimation theorems for Dirichlet boundary conditions 301

4.3 Special cases and examples 305
4.4 More general operators and other boundary conditions 312
4.5 First order initial value problems 316
5 Existence and estimation by norm bounds 323
5.1 The basic theorem 324
5.2 Simultaneous estimation of u^* and $(u^*)'$ 326
5.3 Quadratic growth restrictions 329
6 More general estimates 332
6.1 Estimation by two-sided bounds and norm bounds 333
6.2 Estimates by Lyapunov functions 335
Notes 340

References 343

Notation index 359
Subject index 361