

---

# Contents

<b>List of symbols</b> . . . . .	17
General symbols . . . . .	17
Symbols defined in the text . . . . .	17
<b>Introduction</b> . . . . .	19
<b>Part I. Selfadjoint operators in Hilbert spaces</b> . . . . .	25
<b>1. Preliminaries</b> . . . . .	26
1.1. Fundamental concepts . . . . .	26
1.1.1. Banach spaces and Hilbert spaces . . . . .	26
1.1.2. Borel measures and integrals . . . . .	27
1.1.3. Lebesgue decomposition of Borel measures . . . . .	29
1.2. Vector-valued functions and vector measures . . . . .	29
1.2.1. Borel functions, measurable functions and other types of functions. . . . .	29
1.2.2. Bochner integral . . . . .	30
1.2.3. Banach and Hilbert spaces of vector-valued functions . . . . .	30
1.2.4. Vector measures . . . . .	32
1.2.5. Vector-valued Cauchy-Stieltjes integrals . . . . .	33
1.3. Linear operators. . . . .	35
1.3.1. Elementary properties . . . . .	35
1.3.2. Operator-valued functions . . . . .	35
1.3.3. Selfadjoint, unitary and other special operators . . . . .	35
1.3.4. Compact operators. . . . .	36
1.3.5. Spectra and resolvents . . . . .	38
1.3.6. DUNFORD's functional calculus . . . . .	38
1.4. $C^*$ - and $W^*$ -algebras . . . . .	39
1.4.1. $*$ -algebras . . . . .	39
1.4.2. Commutant and equivalence of projections . . . . .	40
<b>2. Multiplicity theory</b> . . . . .	41
2.1. Simple algebras and algebras with multiplicity . . . . .	41
2.1.1. Simple algebras, simple and disjoint projections . . . . .	41
2.1.2. Central covers. . . . .	42
2.1.3. Algebras and projections with multiplicity . . . . .	42

2.2.	Algebras of type I, decomposition theorem . . . . .	43
2.2.1.	Algebras of type I . . . . .	43
2.2.2.	Abelian v. Neumann algebras . . . . .	43
<b>3.</b>	<b>Spectral theory . . . . .</b>	<b>45</b>
3.1.	Spectral measures, scalar spectral integrals and the spectral theorem . . . . .	45
3.1.1.	Spectral measures on Borel spaces and the Banach space $L^\infty(\Gamma, E)$ . . . . .	45
3.1.2.	The spectral integral and the v. Neumann algebra $T(L^\infty)$ . . . . .	45
3.1.3.	The spectral theorem for bounded normal operators . . . . .	46
3.1.4.	The spectral theorem and the functional calculus for unbounded selfadjoint operators . . . . .	47
3.1.5.	Spectral integrals and Bochner integrals . . . . .	48
3.1.6.	Spectral measures and resolvents . . . . .	49
3.2.	Unitary evolution groups . . . . .	50
3.2.1.	Stone's theorem . . . . .	50
3.2.2.	Operator Fourier transformations . . . . .	50
3.2.3.	The v. Neumann algebra generated by $e^{-itH}$ . . . . .	51
3.3.	Classification of selfadjoint operators and their spectra by properties of their spectral measures . . . . .	52
3.3.1.	The spectrally discrete subspace . . . . .	52
3.3.2.	The spectrally continuous subspace . . . . .	52
3.3.3.	The singularly continuous subspace . . . . .	53
3.3.4.	The absolutely continuous subspace . . . . .	53
3.3.5.	The essential spectrum . . . . .	54
3.3.6.	The spectral core . . . . .	54
3.4.	Spectral properties of evolution groups and resolvents, and "sandwiched" resolvents . . . . .	55
3.4.1.	Characterization of the spectrally discrete and the spectrally continuous subspaces . . . . .	55
3.4.2.	Convergence properties of the resolvent . . . . .	56
3.4.3.	The "sandwiched" spectral measure . . . . .	56
3.4.4.	The "sandwiched" resolvent . . . . .	57
3.5.	Properties of the absolutely continuous subspace . . . . .	58
3.5.1.	$L^1$ -properties and implications . . . . .	58
3.5.2.	Some $L^\infty$ - and $L^p$ -properties ( $p > 1$ ) . . . . .	59
3.5.3.	$L^2$ -properties . . . . .	61
3.5.4.	The absolutely continuous subspace for functions $\alpha(H)$ . . . . .	64
3.5.5.	Spectral manifolds . . . . .	65
<b>4.</b>	<b>Direct integrals and spectral representations . . . . .</b>	<b>67</b>
4.1.	Direct and maximal integrals . . . . .	67
4.1.1.	Preliminaries . . . . .	67
4.1.2.	Direct integrals . . . . .	67
4.1.3.	Maximal integrals . . . . .	68

---

4.2.	Direct sums and tensor products . . . . .	69
4.3.	Direct integrals for separable admissible systems . . . . .	70
4.3.1.	Generating sequences of functions . . . . .	70
4.3.2.	Some properties of direct integrals with separable admissible systems . . . . .	70
4.4.	The algebra of scalar operators and its commutant. . . . .	72
4.4.1.	Scalar operators . . . . .	72
4.4.2.	The commutant of the scalar operators . . . . .	73
4.4.3.	A special case . . . . .	74
4.5.	Unitary invariants and spectral representations . . . . .	75
4.5.1.	Spectral forms, the corresponding direct integrals, and spectral representations . . . . .	75
4.5.2.	Simple operators . . . . .	76
4.5.3.	Unitary invariants of an arbitrary bounded selfadjoint operator and the corresponding spectral representations . . . . .	77
5.	<b>Operator spectral integrals.</b> . . . . .	80
5.1.	Operator spectral integrals defined by Riemann-Stieltjes sums. . . . .	80
5.1.1.	Definitions and connections to scalar spectral integrals . . . . .	80
5.1.2.	Existence of operator spectral integrals . . . . .	81
5.1.3.	Integration by parts . . . . .	82
5.1.4.	An intertwining formula . . . . .	83
5.2.	Vector spectral integrals for absolutely continuous spectral measures . . . . .	85
5.2.1.	Step functions . . . . .	85
5.2.2.	A general class $\mathcal{F}$ of functions . . . . .	86
5.2.3.	Vector spectral integral for functions from $\mathcal{F}$ . . . . .	87
5.2.4.	A spectral representation by vector spectral integrals . . . . .	90
5.2.5.	Vector spectral integrals and Hilbert-Schmidt operators . . . . .	92
5.3.	Approximate operator spectral integrals . . . . .	92
	<b>Notes and remarks to part I</b> . . . . .	93
	<b>Part II. Algebras of asymptotic constants</b> . . . . .	95
6.	<b>General theory of asymptotic constants</b> . . . . .	97
6.1.	Special limiting processes for vector- and operator-valued functions . . . . .	97
6.1.1.	A class of equivalent limiting processes . . . . .	97
6.1.2.	Strong, Abelian and absolute Abelian limits . . . . .	101
6.2.	Definitions and general properties . . . . .	104
6.2.1.	The definition of asymptotic constants . . . . .	104
6.2.2.	Simple properties of the wave morphism . . . . .	105
6.2.3.	Some properties of wave ideals . . . . .	108

6.3.	Relations between the algebras $\mathcal{L}_\infty(\mathcal{H})$ , $\mathcal{L}(\mathcal{H})$ , $\text{dom } \mu_a^H$ and $\text{dom } \mu_a^H$ . . .	111
6.3.1.	Some technical criteria . . . . .	111
6.3.2.	Wave ideals and compact operators . . . . .	113
6.3.3.	The difference between $\ker \mu_a^H$ and $\ker \mu_a^H$ . . . . .	114
6.3.4.	Some relations between $\mathcal{L}(\mathcal{H})$ and $\text{dom } \mu^H$ . . . . .	115
6.4.	Topological characterizations of wave algebras . . . . .	116
6.4.1.	Definitions of special topologies . . . . .	116
6.4.2.	A dense subset in $\text{dom } \mu^H$ . . . . .	118
6.4.3.	Compatibility of the algebraic properties with the $\mu^H$ -topology. . . . .	119
6.5.	Generalizations . . . . .	122
6.5.1.	Universal spectral projections . . . . .	122
6.5.2.	Wave algebras with universal spectral projections . . . . .	124
<b>7.</b>	<b>Special classes of asymptotic constants . . . . .</b>	<b>126</b>
7.1.	Projections . . . . .	126
7.1.1.	Spectral projections of selfadjoint asymptotic constants . . . . .	126
7.1.2.	Projections $P$ with $\mu^H(P) \neq \mu^{-H}(P)$ . . . . .	126
7.1.3.	Equivalence relations between projections . . . . .	127
7.2.	Adjoint and partial isometries . . . . .	129
7.2.1.	Adjoint . . . . .	129
7.2.2.	Properties of partial isometries from $\text{dom } \mu^H$ . . . . .	130
7.2.3.	Unitary operators . . . . .	132
7.3.	$C^*$ - and $W^*$ -subalgebras of the wave algebra . . . . .	133
7.3.1.	$\mu^H$ as $*$ -homomorphism on $*$ -subalgebras . . . . .	133
7.3.2.	$\mu^H$ as spatial isomorphism on $W^*$ -subalgebras . . . . .	134
7.3.3.	Functions of asymptotic constants . . . . .	137
7.4.	Classes of asymptotic constants for special generators . . . . .	139
7.4.1.	The map $\Gamma_H$ and the wave ideal $\ker \mu_s^H$ . . . . .	139
7.4.2.	Multiplication operators . . . . .	142
7.4.3.	Integral operators . . . . .	143
7.4.4.	Integral operators with singularities of Cauchy type . . . . .	145
<b>8.</b>	<b>The invariance of wave morphisms and wave algebras . . . . .</b>	<b>149</b>
8.1.	The class of generators of a wave algebra or wave ideal. . . . .	149
8.1.1.	Some classes of generators . . . . .	149
8.1.2.	On the class of generators of the wave algebra $\text{dom } \mu^H$ . . . . .	152
8.1.3.	The class of generators of the wave ideal $\mathcal{J} = \ker \mu_a^H$ . . . . .	153
8.2.	The invariance of the wave morphism $\mu_a^H$ . . . . .	158
8.2.1.	Admissible functions . . . . .	158
8.2.2.	The invariance of the wave morphism $\mu_a^H$ . . . . .	161
8.3.	The invariance of the wave morphism $\mu_b^H$ . . . . .	163
8.3.1.	A no-go theorem . . . . .	163
8.3.2.	Some special classes of operators . . . . .	164

<b>Notes and remarks to part II</b> . . . . .	167
<b>Part III. Two-space wave operators and scattering operators</b> . . . . .	169
<b>9. Elementary Theory of wave and scattering operators</b> . . . . .	170
9.1. Pre-wave operators and their limits . . . . .	170
9.1.1. Time-dependent and stationary pre-wave operators . . . . .	170
9.1.2. Limits of pre-wave operators . . . . .	171
9.1.3. Simple properties of the limit $J_\infty$ of the pre-wave operator . . . . .	173
9.2. Two-space wave operators . . . . .	175
9.2.1. Definition of wave operators with respect to universal spectral projections . . . . .	175
9.2.2. Completeness and semicompleteness . . . . .	177
9.2.3. One-dimensional perturbations . . . . .	178
9.3. Asymptotically equivalent identification operators . . . . .	181
9.3.1. Simple properties of asymptotically equivalent identification operators . . . . .	181
9.3.2. Criteria for completeness and semicompleteness . . . . .	182
9.4. Wave operators and limits of pre-wave operators . . . . .	183
9.4.1. Connection between $W_+$ and $\Omega_+^w$ . . . . .	184
9.4.2. Asymptotic equivalence of $\Omega_+^w$ and $J$ . . . . .	184
9.5. Partially isometric wave operators . . . . .	185
9.5.1. Asymptotically partially isometric identification operators . . . . .	185
9.5.2. Completeness criteria . . . . .	186
9.6. Scattering operator and scattering matrix . . . . .	187
9.6.1. Scattering operator . . . . .	187
9.6.2. Scattering matrix . . . . .	188
9.7. Examples . . . . .	190
9.7.1. Finite-dimensional perturbations . . . . .	190
9.7.2. Special symmetric hyperbolic systems (uniformly propagative systems) . . . . .	192
<b>10. Identification operators</b> . . . . .	195
10.1. Abstract multichannel scattering theory . . . . .	195
10.1.1. Channel projections and channel Hamiltonians . . . . .	195
10.1.2. Reaction channels and asymptotic completeness . . . . .	197
10.1.3. Example: Multiparticle scattering . . . . .	199
10.1.4. Identification operator . . . . .	200
10.1.5. General channel systems . . . . .	202
10.2. Algebraic scattering theory . . . . .	204
10.2.1. Basic ideas and notions . . . . .	204
10.2.2. Algebraic scattering systems . . . . .	205
10.2.3. General properties of algebraic scattering systems . . . . .	207
10.2.4. Algebraic scattering systems and wave operators . . . . .	208
10.2.5. Examples . . . . .	209
10.3. Equations of second order in $d/dt$ . . . . .	213
10.3.1. Reduction to the Schrödinger equation . . . . .	213
10.3.2. Connection between $U(t)$ and $e^{-iLt}$ . . . . .	214

10.3.3.	The first identification operator . . . . .	215
10.3.4.	The second identification operator . . . . .	217
10.4.	Quantum field theory . . . . .	220
10.4.1.	Abstract quantum fields . . . . .	220
10.4.2.	Tensor algebra of test functions, Poincare group, and regular representation	223
10.4.3.	Quantum fields . . . . .	225
10.4.4.	Example: The free scalar field of mass $m_0 > 0$ . . . . .	228
10.4.5.	Identification operators between free and interacting fields . . . . .	230
10.4.6.	Wave operators for quantum fields . . . . .	232
11.	<b>Structural properties of wave and scattering operators.</b> . . . . .	233
11.1.	Structure of wave operators. . . . .	233
11.1.1.	Structure of wave operators for arbitrary universal projections . . . . .	233
11.1.2.	The special cases $P_H = P_H^c$ and $P_H = P_H^{ac}$ . . . . .	234
11.1.3.	Structure of $H_0$ -semicomplete partially isometric wave operators. . . . .	235
11.1.4.	Structure of complete partially isometric wave operators . . . . .	236
11.2.	Structure of scattering operators (the inverse problem of scattering theory)	238
11.2.1.	Asymptotic partial isometry of identification operators . . . . .	238
11.2.2.	Formulation of the inverse problem . . . . .	240
11.2.3.	The key problem . . . . .	240
11.2.4.	Solution of the inverse problem . . . . .	242
11.2.5.	Description of all solutions . . . . .	243
11.2.6.	The inverse problem for multichannel identification operators . . . . .	245
11.3.	The invariance principle for wave operators. . . . .	245
11.3.1.	Formulation of the invariance principle . . . . .	246
11.3.2.	Invariance principle, strong form . . . . .	246
11.3.3.	Invariance principle, weak form . . . . .	247
11.3.4.	A counterexample and piecewise linear functions . . . . .	248
12.	<b>Lax-Phillips evolutions and two-spaces wave operators</b> . . . . .	250
12.1.	Lax-Phillips evolutions. . . . .	250
12.1.1.	Incoming and outgoing subspaces . . . . .	250
12.1.2.	Examples . . . . .	250
12.1.3.	Structure theorem for Lax-Phillips evolutions . . . . .	251
12.1.4.	Spectral representation of Lax-Phillips evolutions . . . . .	254
12.1.5.	The Lax-Phillips scattering operator . . . . .	254
12.2.	Two-space formulation of the Lax-Phillips scattering theory . . . . .	254
12.2.1.	Construction of special incoming and outgoing translation representations	255
12.2.2.	Wave operators and scattering operator . . . . .	256
12.3.	Analytic properties of the Lax-Phillips scattering matrix . . . . .	258
12.3.1.	Analyticity in the upper half plane. . . . .	258
12.3.2.	Analytic continuation of $\hat{S}(z)$ into the lower half plane and the Lax- Phillips semigroup . . . . .	260

<b>13. Stationary theory</b>	<b>264</b>
13.1. The stationary pre-wave operator	264
13.1.1. Integral representations	264
13.1.2. Some properties of the stationary pre-wave operators	267
13.2. Limits of the stationary pre-wave operator	271
13.2.1. Weak limits	271
13.2.2. Strong limits	274
13.3. Stationary theory of the Abel wave operator in terms of resolvents only	276
13.3.1. Stationary characterization of the Abel wave operator	276
13.3.2. The wave matrix	278
13.4. Stationary theory of the Abel wave operator using operator spectral integrals	280
13.4.1. The general case.	281
13.4.2. The case of partially isometric Abel wave operators	281
13.4.3. Convergence to the wave matrix.	283
13.5. A completeness criterion	284
13.5.1. A property in connection with the adjoint of the wave operator	285
13.5.2. Extension of the operator $Z_+$ and formulation of the criterion	289
<b>Notes and remarks to part III</b>	<b>291</b>
<b>Part IV. Existence and completeness of wave operators</b>	<b>299</b>
<b>14. Stationary methods</b>	<b>301</b>
14.1. Stationary methods with auxiliary manifolds	301
14.1.1. Spectral forms and spectral manifolds	301
14.1.2. Approximate spectral forms and auxiliary manifolds	302
14.1.3. Existence and completeness of wave operators	304
14.1.4. Existence of strong wave operators	310
14.2. Factorization method	314
14.2.1. The perturbation $V$ and auxiliary manifolds	314
14.2.2. Existence of wave operators under factorization assumptions	316
14.2.3. Completeness of wave operators under smallness and compactness assumptions.	317
14.3. Applications	319
14.3.1. Multiplication operator perturbed by integral operators (small gentle perturbations).	319
14.3.2. One-dimensional perturbation	321
<b>15. Time-falloff methods</b>	<b>322</b>
15.1. Existence of wave operators	322
15.1.1. Some abstract criteria	322
15.1.2. Application: Short range perturbations of the Laplacian	325

15.1.3.	Application: Schrödinger operators with hard core potential . . . . .	326
15.1.4.	Application: Schrödinger operators with rapidly oscillating potentials . . . . .	327
15.2.	Completeness of wave operators . . . . .	330
15.2.1.	Completeness from spectral assumptions on $H$ . . . . .	330
15.2.2.	Completeness by time-falloff conditions. . . . .	331
15.2.3.	Application: Differential operators of first order . . . . .	337
15.2.4.	Application: Potential scattering . . . . .	338
15.3.	An invariance principle. . . . .	340
<b>16.</b>	<b>Trace class methods . . . . .</b>	<b>343</b>
16.1.	Wave operators for trace class perturbations . . . . .	343
16.1.1.	A general theorem about trace class perturbations . . . . .	343
16.1.2.	Other trace class results . . . . .	346
16.2.	A stationary proof of a trace class theorem . . . . .	347
16.3.	Applications . . . . .	351
16.3.1.	Wave operators for $H = H_1 + H_2$ , $H_0 = H_1 \oplus H_2$ . . . . .	351
16.3.2.	Wave operators for Dirac operators . . . . .	353
16.3.3.	Wave operators for a weakly uniformly propagative system . . . . .	354
<b>17.</b>	<b>Smooth perturbations . . . . .</b>	<b>356</b>
17.1.	Smooth operators . . . . .	356
17.1.1.	Basic notions . . . . .	356
17.1.2.	Representation as integral operators . . . . .	361
17.1.3.	Local smoothness . . . . .	364
17.2.	Existence of wave operators . . . . .	367
17.2.1.	Symmetrical case . . . . .	367
17.2.2.	Smallness condition . . . . .	371
17.3.	Applications . . . . .	373
17.3.1.	Smooth perturbations of $-i(d/dx)$ . . . . .	373
17.3.2.	Potential scattering . . . . .	374
<b>Notes and remarks to part IV . . . . .</b>		<b>374</b>
<b>Part V. Some properties of the scattering operator, the scattering matrix, and the scattering amplitude . . . . .</b>		<b>381</b>
<b>18.</b>	<b>Representations of the scattering operator and the scattering amplitude, and analyticity properties of the scattering amplitude . . . . .</b>	<b>382</b>
18.1.	General formulas and properties for $S$ and $\hat{T}(\lambda)$ within the framework of the stationary theory . . . . .	382
18.1.1.	Scattering amplitude and total scattering cross section . . . . .	382
18.1.2.	A representation of the scattering operator by a spectral integral . . . . .	382
18.1.3.	A general formula for $\hat{T}(\lambda)$ . . . . .	386



18.1.4. An explicit formula for $\hat{T}(\lambda)$ . . . . .	387
18.1.5. An application to potential scattering . . . . .	391
18.2. Smoothness conditions and the scattering amplitude . . . . .	391
18.2.1. Some formulas for the scattering amplitude . . . . .	391
18.2.2. A special case in the one-space theory . . . . .	393
18.3. Analyticity properties of the scattering amplitude and resonances . . . . .	394
18.3.1. General remarks . . . . .	394
18.3.2. Analytic continuations of $\hat{T}(\lambda)$ . . . . .	395
18.3.3. An application to potential scattering . . . . .	397
18.3.4. The finite-dimensional Friedrichs model . . . . .	398
<b>19. Spectral properties of the scattering amplitude . . . . .</b>	<b>400</b>
19.1. Trace class conditions and scattering amplitude . . . . .	400
19.1.1. Calculation of the scattering amplitude for trace class perturbations . . . . .	400
19.1.2. A trace norm estimate of the scattering amplitude . . . . .	402
19.1.3. Example (potential scattering) . . . . .	404
19.1.4. The spectral shift and the trace formula . . . . .	405
19.1.5. Phase shift and spectral shift . . . . .	407
19.2. Time-falloff conditions and scattering amplitude. . . . .	410
19.2.1. Spectral properties of parts of $S - P_0^{ac}$ . . . . .	410
19.2.2. The scattering amplitude for “locally” compact perturbations . . . . .	413
19.2.3. The scattering amplitude for “locally” Hilbert-Schmidt perturbations . . . . .	415
19.2.4. The scattering amplitude for a special scattering system . . . . .	415
<b>Notes and remarks to part V . . . . .</b>	<b>417</b>
<b>Bibliography . . . . .</b>	<b>420</b>
Books and monographs . . . . .	420
Articles . . . . .	422
<b>Subject index . . . . .</b>	<b>447</b>