

Contents

Part I: Fundamentals of Operator Algebras		1
1	Preliminaries	2
1.1	Topological spaces and manifolds	2
1.1.1	Basic notions	2
1.1.2	Locally convex spaces	3
1.1.3	Test function spaces and distributions	4
1.1.4	Manifolds	5
1.2	Banach spaces and Hilbert spaces	6
1.2.1	Banach spaces	6
1.2.2	Vector-valued functions and integrals	6
1.2.3	Hilbert spaces and Banach spaces	7
1.2.4	Linear operators in Hilbert spaces	8
1.2.5	Spectrum and spectral measure	9
1.2.6	The spectral theorem and Stone's theorem	9
1.2.7	Classification of the spectra	10
1.3	Topological groups	10
1.3.1	Basic notions	10
1.3.2	Haar measure	11
1.3.3	Lie groups	11
1.3.4	Continuous unitary representations and the SNAG - theorem	11
1.3.5	Compact groups	12
1.4	Riemannian spaces	12
1.4.1	Riemannian manifolds	12
1.4.2	Examples of Riemannian spaces	13
1.4.3	Test function spaces and distributions on Riemannian spaces	14
1.4.4	Klein - Gordon equation	15
2	C* - Algebras	16
2.1	Fundamental concepts	16
2.1.1	Basic notions	16
2.1.2	Abelian C*-algebras and spectrum	17
2.1.3	Positivity and order relation	18
2.2	Representations and states	18
2.2.1	Representations	18
2.2.2	States	19
2.2.3	States for abelian C*-algebras	20
2.2.4	GNS - construction and universal representation	20
2.3	Tensor products of C*-algebras	22
2.4	Group C*-algebras	23
2.4.1	Basic notions	23
2.4.2	The reduced group C*-algebra	23

2.4.3	Unitary representations of compact groups	23
2.4.4	Characters of compact groups	24
 3	Von Neumann Algebras	25
3.1	Von Neumann algebras and their preduals	25
3.1.1	Topologies on $\mathcal{L}(\mathcal{H})$ and the double commutant theorem	25
3.1.2	Definition of von Neumann algebras and simple properties	26
3.1.3	The predual of $\mathcal{L}(\mathcal{H})$	26
3.1.4	The predual of a von Neumann algebra	27
3.2	Normal states and representations	28
3.2.1	Normal states	28
3.2.2	Representations	28
3.3	Projections and classification	29
3.3.1	Traces and weights	29
3.3.2	Projections	30
3.4	Tensor products and decomposition of von Neumann algebras	31
3.5	Modular theory, a first step	32
 4	 C^* - Systems and von Neumann Systems	34
4.1	C^* -systems and elementary spectral theory	34
4.1.1	C^* -systems	34
4.1.2	The mean projection	35
4.1.3	Further spectral projections	35
4.2	\mathcal{G} -invariant states and states on the fixed point algebra	36
4.2.1	\mathcal{G} -invariant states and \mathcal{G} -ergodic states	36
4.2.2	Simple properties of \mathcal{G} -ergodic states	38
4.2.3	States on $\mathcal{F}^\mathcal{G}$ and \mathcal{G} -invariant states on \mathcal{F}	38
4.2.4	Vacuum representations of \mathcal{A} , in particular for \mathcal{G} -ergodic states	39
4.3	C^* -crossed products	40
4.3.1	The Banach $*$ -algebra $L^1(\mathcal{G}, dg; \mathcal{F})$	40
4.3.2	Covariant representations of C^* -systems	41
4.3.3	Connection between covariant representations of $(\mathcal{F}, \alpha_{\mathcal{G}})$ and non-degenerate $*$ -representations of $\mathcal{F} \otimes_{\alpha} \mathcal{G}$	42
4.3.4	A special case of the bijection theorem	43
4.3.5	Reduced crossed products	44
4.4	Assigned von Neumann systems	44
4.4.1	The central decomposition	44
4.4.2	The case $\mathcal{Z}_{\mathcal{G}} \subseteq \mathcal{Z}_{\mathcal{A}}$	45
Notes and Remarks to Part I	46	
 Part II: Causal Nets of Operator Algebras	48	
5	Basic Theory of Nets of C^* -algebras and von Neumann Algebras	49
5.1	Nets of C^* - and von Neumann algebras; quasilocal algebras	49
5.1.1	Net index sets	49
5.1.2	Nets of $*$ -algebras, C^* -algebras, and von Neumann algebras	51

5.1.3 GNS-construction and nets of von Neumann algebras	53
5.1.4 Tensor products of nets of C^* -algebras and von Neumann algebras	54
5.2 Locally normal linear forms	54
5.2.1 Simple properties	54
5.2.2 The characteristic locally convex topology \mathcal{T}	57
5.2.3 Existence of sufficiently many locally normal states	57
5.2.4 A locally uniform topology for $\mathcal{F}_{ln}^\#$	59
5.3 Representations of nets of algebras	59
5.3.1 Relative commutant and compatibility	59
5.3.2 Relative commutant for local algebras; Commutant compatibility (CC) of representations	60
5.3.3 Relative bicommutant	62
5.3.4 Cluster properties of states	62
5.3.5 Simplicity of the quasilocal algebra	64
5.4 Split and quasi-split property	65
5.4.1 Simple properties	65
5.4.2 Quasi-split property and relative bicommutant property	66
5.4.3 Split property for nets of von Neumann algebras	67
5.5 Reeh-Schlieder property and some simple consequences	67
5.5.1 Reeh-Schlieder property	67
5.5.2 Full correlation of states	67
 6 Nets of C^* -Systems and von Neumann Systems	70
6.1 C^* -systems $(\mathcal{F}, \alpha_{\mathcal{G}})$ for quasilocal algebras \mathcal{F} , compatible with the net structure of \mathcal{F}	70
6.1.1 Basic concepts	70
6.1.2 Nets of local fixed point algebras	71
6.2 Special automorphism groups for nets of local algebras	71
6.2.1 Net compatible automorphisms and gauge transformations	71
6.2.2 Automorphisms commuting with the gauge group	72
6.3 Crossed products of nets of C^* -systems	72
6.3.1 Crossed products of nets	72
6.3.2 First connection between fixed point nets and crossed product nets	73
6.3.3 Second connection between fixed point nets and crossed product nets	73
6.4 Standard inclusions and standard nets	74
6.4.1 Standard states, standard vectors of von Neumann inclusions	74
6.4.2 Standard states and standard vectors for von Neumann nets; standard nets	75
6.4.3 Standard form of a von Neumann algebra \mathcal{N} , standard cone and standard implementation of $\text{aut } \mathcal{N}$	76
6.4.4 The universal map Φ_A for standard split inclusions	77
6.4.5 Automorphisms of standard nets	78
6.4.6 Further properties of standard split nets of von Neumann algebras; Schlieder property	79
6.4.7 Net compatible automorphisms commuting with the gauge group	80
6.5 Local implementation of net automorphisms	80
6.5.1 Sequences of standard split inclusions	80

6.5.2 Local implementation of net automorphisms	81
7 Causal Nets of C*-Algebras and von Neumann Algebras	82
7.1 Causal disjointness and causal complement	82
7.1.1 Causal disjointness	82
7.1.2 Causal complement	83
7.1.3 Causal complements and special sets	86
7.1.4 Causal topological spaces	88
7.2 Nets of C*-algebras and von Neumann algebras with causality condition	94
7.2.1 Basic concepts and first examples	94
7.2.2 Causal disjointness relation determined by a given net	95
7.2.3 Cluster properties of states; factorizing states. Theorem of Lanford-Ruelle	96
7.2.4 Duality	99
7.2.5 Causality condition and covariance	102
7.3 Causal nets over causal topological spaces	103
7.3.1 Basic properties, extensions, additivity, tensor products	104
7.3.2 Causal dynamical law, primitive causality, weak additivity	105
7.3.3 Simple implications; Wightman's inequality	106
7.3.4 Extended locality; the centers of local algebras	107
7.3.5 Simplicity of \mathcal{A} , a property of α_σ	108
7.3.6 Simple causal nets of algebras	109
7.3.7 Causal nets and spectrality	112
8 CAR - Algebras and CCR - Algebras	120
8.1 CAR - algebras	120
8.1.1 Canonical anticommutation relations (CAR)	120
8.1.2 Preliminaries	121
8.1.3 The spin algebra	121
8.1.4 Jordan-Wigner transformations	122
8.1.5 The C*-norm for the generating elements $A(h), h \in \mathcal{K}$	123
8.1.6 The case $\dim \mathcal{K} = 2n, n < \infty$	124
8.1.7 Construction of an algebra \mathcal{A}_σ for an infinite-dimensional Hilbert space \mathcal{K}	124
8.1.8 The case (\mathcal{K}, Γ) where \mathcal{K} is an infinite-dimensional Hilbert space	125
8.1.9 The case $\dim \mathcal{K} < \infty$ and odd	126
8.1.10 Creation and annihilation operators; basis projections	127
8.1.11 Fock states on $CAR(\mathcal{K}, \Gamma)$	127
8.1.12 Bogoliubov automorphisms	130
8.1.13 Net structures within $CAR(\mathcal{K}, \Gamma)$	131
8.2 CCR - Algebras	132
8.2.1 Canonical commutation relations (CCR)	132
8.2.2 Uniqueness result, Bogoliubov automorphisms	132
8.2.3 Regular states of $CCR(H, \sigma)$	133
8.2.4 Fock states	135

8.2.5 Number operator, Stone- v.Neumann uniqueness theorem	136
8.2.6 Unitary implementation of symplectic transformations	137
8.2.7 Unitarily invariant (regular) states of $\text{CCR}(H, \sigma)$	139
8.2.8 Net structures of $\text{CCR}(H, \sigma)$	140
8.3 Nets with causality condition on globally hyperbolic space-times connected with net structures of $\text{CAR}(H, \Gamma)$ or $\text{CCR}(H, \sigma)$	140
Notes and Remarks to Part II	144

Part III: Superselection Theory	150
9 Basic Structural Elements of Superselection Theory Deduced from a Selection Principle for Admissible Representations of Quasilocal Algebras	152
9.1 Basic assumptions on the causal nets of algebras	152
9.2 Admissible representations and their basic properties	154
9.2.1 Definitions	154
9.2.2 Properties of admissible representations	156
9.3 Permutation symmetry	157
9.3.1 Intertwiners	158
9.3.2 Cross products of intertwiners	158
9.3.3 Causally disjoint intertwiners	158
9.3.4 The permutation operator $\epsilon(\rho, \sigma)$	160
9.4 The unitary representation $\epsilon_\rho^{(n)}(P_n)$	161
9.4.1 Intertwiners for a finite number of endomorphisms	161
9.4.2 The case of identical endomorphisms	162
9.5 Left inverses for admissible endomorphisms	163
9.5.1 Admissible endomorphisms as limits of sequences of admissible inner automorphisms	163
9.5.2 Existence of left inverses for admissible endomorphisms	164
9.5.3 Properties of left inverses	165
9.5.4 Spectral properties of $\Phi(\epsilon_\rho)$	166
9.6 Admissible endomorphisms ρ and corresponding left inverses Φ with $\Phi(\epsilon_\rho) = \lambda 1$	167
9.6.1 The case $\lambda = 0$	167
9.6.2 The case $\lambda \neq 0$	168
9.6.3 Classification by λ	168
9.7 Left inverses for direct sums, subobjects and products	169
9.7.1 A preliminary result	169
9.7.2 Direct sums	170
9.7.3 Subobjects	170
9.7.4 Products	171
9.8 Finite endomorphisms and invariants	172
9.8.1 Finite endomorphisms	172
9.8.2 Invariants for finite endomorphisms	173
9.9 Standard left inverses	175
9.9.1 Definitions and simple properties	175
9.9.2 Description of all left inverses for finite endomorphisms	177
9.9.3 Bosonic and fermionic endomorphisms	179

9.10	Conjugated endomorphisms and conjugates	179
9.10.1	Preliminary results and definitions	179
9.10.2	The pure bosonic case	182
9.10.3	The pure fermionic case	183
9.10.4	Composition of the pure bosonic and the pure fermionic case	183
9.11	Bosonization	183
9.11.1	Preparing remarks	184
9.11.2	The modified permutation operator	185
9.11.3	Bosonization	186
10	Hilbert C*-Systems for Compact Groups; Canonical Representations of the Fixed Point Algebra and Superselection Structure	189
10.1	Hilbert C*-systems	189
10.1.1	Definitions and basic properties	189
10.1.2	Canonical endomorphisms	192
10.1.3	Link between center, relative commutant and special properties of the canonical endomorphisms	193
10.2	The subalgebra \mathcal{F}_{fin}	195
10.2.1	The conjugated representation \tilde{D}	195
10.2.2	The products $\Phi_{D,s}\Phi_{D',s}$	196
10.2.3	A special subalgebra	196
10.3	\mathcal{A} -module isomorphic Hilbert systems	198
10.3.1	The algebra \mathcal{F} as an extension of \mathcal{A}	198
10.3.2	Products, cocycle relation for the corresponding coefficient scheme, equivalence relation	199
10.3.3	Permutation operator	200
10.4	Construction of Hilbert systems by crossed products	204
10.4.1	\mathcal{G} -duals	205
10.4.2	Conjugated arrows	205
10.4.3	The categories $\text{End } \mathcal{A}$ and $\text{End}_1 \mathcal{A}$	206
10.4.4	Actions of \mathcal{G} -duals on \mathcal{A} ; functors from the \mathcal{G} -dual into the category $\text{End}_1 \mathcal{A}$	207
10.4.5	The \mathcal{A} -left module \mathcal{F}_o	208
10.4.6	The assignment \tilde{T}	209
10.4.7	The bimodule property	210
10.4.8	The product structure in \mathcal{F}_o	211
10.4.9	The *-algebra \mathcal{F}_o	212
10.4.10	Algebraic Hilbert spaces in \mathcal{F}_o	215
10.4.11	A C*-norm on the *-algebra \mathcal{F}_o	217
10.5	Hilbert C*-systems; continuation	219
10.5.1	Algebraic Hilbert spaces in (\mathcal{F}, α_g)	219
10.5.2	The category of all finite-dimensional invariant algebraic Hilbert spaces in \mathcal{F} (w.r.t. 1 and with support 1)	222
10.5.3	The ‘canonical’ functor T from $\mathcal{D}(\mathcal{F})$ into $\text{End}_1 \mathcal{A}$	223
10.5.4	The subcategory of $\text{End}_1 \mathcal{A}$ defined by T	223
10.5.5	The stabilizer of \mathcal{A}	225
10.5.6	Flip isomorphism and permutation symmetry	226

10.5.7	The permutation $\Theta_{\mathcal{H}}$	228
10.5.8	The unitary representation $\Theta_{\mathcal{H}}^n(\pi)$, $\pi \in P_n$, of the permutation group P_n , $n = 1, 2, \dots$	228
10.5.9	The special element S and its properties	230
10.5.10	The conjugated representations and their Hilbert spaces	230
10.5.11	Conjugated elements	231
10.5.12	Left inverses	232
10.5.13	Conditional expectations	233
10.5.14	Further properties of $\Phi_{\mathcal{H}}$	234
10.5.15	The iterates $\Phi_{\mathcal{H}}^n$ of $\Phi_{\mathcal{H}}$ as faithful numerical traces	235
10.5.16	Properties of the representation $\Theta_{\mathcal{H}}^n(\cdot)$ of P_n	237
10.6	Hilbert -v.-Neumann systems; canonical states of \mathcal{A}	238
10.6.1	\mathcal{G} -invariant states on $(\mathcal{F}, \alpha_{\mathcal{G}}; \mathcal{A})$	238
10.6.2	Hilbert -v.-Neumann systems	242
10.6.3	Simple properties of Hilbert -v.-Neumann systems	243
10.6.4	Hilbert -v.-Neumann systems of special type	245
10.6.5	Canonical representations of $\mathcal{A} = \Pi, \mathcal{F}$	251
11	Inverse Superselection Theory: Determination of the Gauge Group and the Field Algebra from the Superselection Structure	253
11.1	Elementary theory of the category \mathcal{T}	254
11.1.1	The category $\text{End } \mathcal{A}$	254
11.1.2	Unitary equivalence, disjointness, irreducibility	254
11.1.3	The axioms of \mathcal{T}	254
11.1.4	Simple properties of permutation unitaries, conjugation and conjugates	255
11.1.5	Transfer of the conjugation to the arrows	257
11.1.6	The dimension $d(\rho)$ of $\rho \in \text{Ob } \mathcal{T}$	259
11.1.7	Further properties of the permutation unitaries	260
11.1.8	The left inverses Ψ_{ρ} and Φ_{ρ}	261
11.1.9	Left inverses for subobjects	265
11.1.10	Decomposition properties of objects	266
11.1.11	The determinant	268
11.1.12	Special objects, dominance and construction of conjugates for special objects	271
11.2	Basic material on the Cuntz algebra \mathcal{C}	274
11.2.1	Basic definitions	274
11.2.2	The algebraic part \mathcal{C}^o of \mathcal{C}_d and its graduation structure	275
11.2.3	The subalgebra $\mathcal{C}^{o,0}$	276
11.2.4	Representation of S^1 by automorphisms of \mathcal{C}^o	277
11.2.5	Simplicity of \mathcal{C}	278
11.2.6	The canonical representation of $U(d)$ in $\text{aut } \mathcal{C}$	278
11.2.7	The fixed point algebra $\mathcal{C}_{U(d)}$ and its generating elements	279
11.2.8	The fixed point algebra $\mathcal{C}_{SU(d)}$ and its generating elements	279
11.2.9	Simplicity of $\mathcal{C}_{SU(d)}$	280
11.3	Tensor products and crossed products of \mathcal{A} and \mathcal{C}_d w.r.t. a special $\rho \in \text{Ob } \mathcal{T}$ and w.r.t. $\mathcal{C}_{SU(d)}$	281
11.3.1	Embedding of $\mathcal{C}_{SU(d)}$ into \mathcal{A}	281

11.3.2 The $(C^o)_{SU(d)}$ -module $\mathcal{A} \otimes_\mu C_d^o$	282
11.3.3 The $(C^o)_{SU(d)}$ -module $\mathcal{A} \otimes_\mu C_d^o$ as an algebra	282
11.3.4 The algebra $\mathcal{A} \otimes_\mu C_d^o$ as a *-algebra	284
11.3.5 An intermediate step for introducing a C^* -norm	286
11.3.6 Application of Section 5 to the present case	287
11.3.7 Summary	288
11.3.8 Coincidence of relative commutant and center	288
11.3.9 Ergodicity of $\hat{\alpha}_G$ on the center \mathcal{Z} of \mathcal{B} and transitivity of the transposed action on $\text{spec } \mathcal{Z}$	289
11.3.10 Stability subgroups	290
11.4 Derived C^* -systems with trivial relative commutant (Mackey theory)	290
11.4.1 Representations of \mathcal{B} governed by elements of $\text{spec } \mathcal{Z}$	290
11.4.2 Reconstruction of (\mathcal{B}, α_G) from $(\mathcal{B}^\Phi, \alpha_{\mathcal{G}_+}^\Phi)$	292
11.4.3 Uniqueness result	294
11.4.4 Summary	294
11.4.5 Uniqueness of the C^* -systems (\mathcal{B}, β_Q)	295
11.5 Realization of the category \mathcal{T} on \mathcal{A} by a Hilbert space category, extending \mathcal{A} to a Hilbert C^* -system w.r.t. a compact group	297
11.5.1 Main results	297
11.5.2 Proof of Theorem 1 and Corollary 2	300
11.5.3 Proof of Theorem 3	302
11.5.4 Elementary concepts of Galois theory	303
11.5.5 Summary	304
11.6 Special case, remarks, problems and application	305
11.6.1 Abelian group \mathcal{G}	305
11.6.2 An inverse problem	306
11.6.3 Application to a quasilocal algebra \mathcal{A} of a causal net	306
Notes and Remarks to Part III	307
Part IV: Causal Nets of Algebras and Quantum Fields	314
12 Quantum Fields	315
12.1 Quantum fields and complete sets of quantum fields	315
12.1.1 Quantum fields	315
12.1.2 Complete sets of quantum fields	316
12.2 Test function algebra, identification operator, and Wightman functional	317
12.2.1 Test function algebra and identification operator	317
12.2.2 Wightman functional	320
12.2.3 Spectrality and identification operator	322
12.3 Consequences of spectrality assumptions	326
12.4 Quantum fields as pointlike localized objects	327
12.4.1 Transitive Lie groups of isometries	327
12.4.2 Quantum fields and special dense sets	328
12.4.3 Quantum fields at a point	331
12.4.4 Energetic-bounded quantum fields	334

12.5	Linear quantum fields on globally hyperbolic manifolds	337
12.5.1	A general construction	337
12.5.2	The Minkowski space-time	339
12.5.3	A second construction starting with a Cauchy surface of \mathcal{X}	340
12.5.4	Minkowski space-time, revisited	343
13	Causal Nets of von Neumann Algebras for Complete Sets of Quantum Fields	345
13.1	Preliminaries	345
13.1.1	Sets of unbounded operators	345
13.1.2	Quasianalytic vectors and commutation relations	347
13.1.3	Linear energetic bounds	349
13.2	Definitions and properties of causal nets for quantum fields	351
13.2.1	General definitions	351
13.2.2	Causality condition under additional assumptions	354
13.3	Causal nets for manifolds with wedge regions	356
13.3.1	Complex Lorentz-transformations and field operators	356
13.3.2	Special duality for quantum fields	359
13.3.3	Special duality for causal nets of von Neumann algebras	361
13.3.4	Additivity for causal nets of von Neumann algebras	366
14	Quantum Fields from Causal Nets of Algebras	368
14.1	Reconstruction of quantum fields	368
14.1.1	Operator-valued functions and distributions	368
14.1.2	Reconstruction of quantum fields	369
14.2	Construction of all quantum fields	371
14.2.1	Topologies for the causal net of von Neumann algebras	371
14.2.2	Construction of all quantum fields	372
14.3	Energetic-bounded quantum fields	373
14.3.1	Preliminaries	373
14.3.2	Energetic-bounded quantum fields	375
14.3.3	Generating vectors for quantum fields	379
Notes and Remarks to Part IV	381	
 Part V: Modular Theory, Type Theory and Injectivity of Local Algebras		
15	Modular Theory and Type III - Factors	384
15.1	Modular theory	385
15.1.1	Notations and simple examples	385
15.1.2	Modular theory for finite-dimensional algebras	386
15.1.3	Modular theory	387
15.1.4	KMS - boundary conditions	390
15.1.5	The unitary cocycle theorem	392
15.2	Classification of type III - factors	394
15.2.1	Arveson spectrum	394
15.2.2	Connes spectrum and classification	396
15.2.3	Criteria for the type III - property	399

16	Modular Theory and Type Theory for Local Algebras	402
16.1	Modular theory for local algebras	402
16.1.1	Local algebras for wedge regions	402
16.1.2	Asymptotic comparison for modular operators	402
16.1.3	Asymptotic comparison of modular conjugations	406
16.1.4	Existence of quantum fields	408
16.2	Type of local algebras	409
16.2.1	Some relations between the types of an algebra and a subalgebra	409
16.2.2	Local algebras of type III	415
16.2.3	Local algebras of type III_1	416
16.2.4	Scaling limit and type III_1 for local algebras	417
17	Nuclearity and Injectivity of Local Algebras	421
17.1	Nuclearity and split property	421
17.1.1	Nuclear maps	421
17.1.2	Nuclearity and split property	424
17.2	The universal structure of local algebras	426
17.3	Nuclearity for free causal nets over Minkowski space-time	427
	Notes and Remarks to Part V	431
	Bibliography	433
	Index	453