Contents

Preface *v*

A Note on Method *vii*

Chapter 1. First Order Linear Differential Equations **1**

1.1. The Equation $a(x)y' + b(x)y = h(x)$ 1
1.2. First Order Linear Differential Expressions; the Kernel 4
1.3. Finding a Particular Solution by Variation of Parameters 9
1.4. Power Series Review 10
1.5. The Initial Value Problem for a First Order Linear Differential Equation 18

Chapter 2. Second Order Linear Differential Equations **22**

2.1. Basic Concepts of Linear Algebra for Function Spaces 22
2.2. The Initial Value Problem for a Second Order Linear Homogeneous Differential Equation 28
2.3. Dimension of the Kernel; General Solution; Abel's Formula 35
2.4. Kernel of Constant-Coefficient Expressions 39
2.5. The Classical Linear Oscillator 43
2.6. Guessing a Particular Solution to a Constant-Coefficient Equation 48
2.7. Particular Solution by Variation of Parameters 51
2.8. The Kernel of Legendre's Differential Expression 53

2.9. The Kernels of Other Classical Expressions 57
2.10. Dirac's Delta Function and Green's Functions 59
Appendix 2.A. Second Order Linear Differential Equations in the Complex Domain 69

Chapter 3. Hilbert Space 78
3.1. The Vibrating Wire 78
3.2. Fourier Series 83
3.3. Fourier Sine and Cosine Series 94
3.4. Fourier Series over Other Intervals 99
3.5. The Vibrating Wire, Revisited 102
3.6. The Inner Product 108
3.7. Schwarz's Inequality 118
3.8. The Mean-Square Metric; Orthogonal Bases 127
3.9. L^2 Spaces 138
3.10. Hilbert Space 149

Chapter 4. Linear Second Order Differential Operators in L^2 Spaces and Their Eigenvalues and Eigenfunctions 157
4.1. Compatibility 157
4.2. Eigenvalues and Eigenfunctions 168
4.3. Hermitian Operators 175
4.4. Some General Operator Theory 184
4.5. The One-Dimensional Laplacian 191
4.6. Legendre's Operator and Its Eigenfunctions, the Legendre Polynomials 202
4.7. Solving Operator Equations with Legendre's Operator 217
4.8. More on Legendre Polynomials: Rodrigues' Formula, the Recursion Relation, and the Generating Function 224
4.9. Hermite's Operator and Its Eigenfunctions, the Hermite Polynomials 233
4.10. Solving Operator Equations with Hermite's Operator 243
4.11. More on Hermite Polynomials: Rodrigues' Formula, the Recursion Relation, and the Generating Function 248
Appendix 4.A. Mathematical Aspects of Differential Operators in L^2 Spaces 253

Chapter 5. Schrödinger's Equations in One Dimension 267
5.1. The Wave Equation by the Hilbert Space Method 267
5.2. The Heat Equation by the Hilbert Space Method 273
5.3. Quanta as Eigenvalues—the Time-Independent Schrödinger Equation in One Dimension 282

5.4. Interpretation of the Ψ Function. The Time-Dependent Schrödinger Equation in One Dimension 289
5.5. The Quantum Linear Oscillator 298
5.6. Solution of the Time-Dependent Schrödinger Equation 308
5.7. A Brief History of Matrix Mechanics 313
5.8. A General Formulation of Quantum Mechanics: States 316
5.9. A General Formulation of Quantum Mechanics: Observables 326

Chapter 6. Bessel's Operator and Bessel Functions 331

6.1. The Wave Equation and Other Equations in Higher Dimensions; Polar Coordinates 331
6.2. Bessel's Equation and Bessel's Operator of Order Zero 339
6.3. $J_0(x)$: The Bessel Function of the First Kind of Order Zero 345
6.4. $J_0(x)$: Calculating Its Values and Finding Its Zeros 350
6.5. The Eigenvalues and Eigenfunctions of Bessel's Operator of Order Zero 359
6.6. The Vibrating Drumhead, the Heated Disk, and the Quantum Particle Confined to a Circular Region (the θ-Independent Case) 367
6.7. θ Dependence: Bessel's Equation and Bessel's Operator of Integral Order p 375
6.8. $J_p(x)$: The Bessel Functions of the First Kind of Integral Order p 380
6.9. The Eigenvalues and Eigenfunctions of Bessel's Operator of Integral Order p 387
6.10. Project on Bessel Functions of Nonintegral Order 392
Appendix 6.A. Mathematical Theory of Bessel's Operator 393

Chapter 7. Eigenvalues of the Laplacian, with Applications 402

7.1. The Laplacian as a Hilbert Space Operator 402
7.2. Differential Forms, the Stokes Theorem, and Integration by Parts in Two Variables 410
7.3. The Laplacian Is Hermitian 418
7.4. General Facts About the Eigenvalues of the Laplacian 426
7.5. Eigenvalues of the Rectangle 440
7.6. Eigenvalues of the Disk 450
7.7. The Laplacian on the Sphere 452
7.8. Eigenvalues of the Sphere; Spherical Harmonics 466
7.9. The Hydrogen Atom 481
7.10. Project on Laguerre's Operator and Laguerre Polynomials 493

7.11. Laplace's Equation and Harmonic Polynomials 500
Appendix 7.A. The Legendre, Laguerre, and Schrödinger Operators 507

Chapter 8. The Fourier Transform **514**

8.1. Complex Methods in Fourier Series; the Fourier Transform 514
8.2. Plancherel's Theorem; Examples of Fourier Transforms 520
8.3. Fourier Sine and Cosine Transforms 526
8.4. The Fourier Transform Is a Unitary Operator on $L^2(-\infty, \infty)$. 529
8.5. The Fourier Transform Converts Differentiation into Multiplication by the Independent Variable 537
8.6. The Eigenvalues and Eigenfunctions of the Fourier Transform 544
Appendix 8.A. $\int_{-\infty}^{\infty} ((\sin x)/x)\, dx = \pi$ 551

Index of Symbols *553*

Index *555*