Contents

Preface and Acknowledgments xv

1 Markov Chains 3

1.1 Introduction . . . 3
1.2 Markov Chains . . . 4
1.3 Discrete-Time Markov Chains . . . 5
1.3.1 Definition . . . 5
1.3.2 The Chapman–Kolmogorov Equations . . . 6
1.3.3 Classification of States . . . 8
1.3.4 Irreducibility . . . 13
1.3.5 Probability Distributions . . . 14
1.3.6 Steady-State Distributions of Ergodic Markov Chains . . . 16
1.4 Continuous-Time Markov Chains . . . 17
1.4.1 Definitions . . . 17
1.4.2 Transition Probabilities and Transition Rates . . . 18
1.4.3 The Embedded Markov Chain . . . 20
1.4.4 The Chapman–Kolmogorov Equations . . . 22
1.4.5 Probability Distributions . . . 22
1.5 Nonnegative Matrices . . . 24
1.5.1 Definition . . . 24
1.5.2 Nonnegative Decomposable Matrices . . . 25
1.5.3 The Theorem of Perron–Frobenius . . . 26
1.6 Stochastic Matrices . . . 28
1.6.1 Definition . . . 28
1.6.2 Some Properties of Stochastic Matrices . . . 28
1.6.3 Effect of the Discretization Parameter, Δt . . . 31
1.6.4 Eigenvalues of Decomposable Stochastic Matrices . . . 33
1.6.5 Eigenvectors of Decomposable Stochastic Matrices . . . 35
1.6.6 Nearly Decomposable Stochastic Matrices . . . 38
1.7 Cyclic Stochastic Matrices . . . 39
1.7.1 Definition . . . 39
1.7.2 Eigenvalues of Cyclic Stochastic Matrices . . . 40

1.7.3 Example: A Decomposable System with Cyclic Subgroups 40
1.8 Indicators of State Clustering 42
1.8.1 Significance of Subdominant, Right-Hand Eigenvectors 42
1.8.2 A Resource Pool Example 44
1.9 Queueing Models 49
1.9.1 Specification 50
1.9.2 Markov Chain Analysis of Queueing Networks 53
1.9.3 Example 1: Two Stations in Tandem 54
1.9.4 Example 2: One Coxian and Two Exponential Servers 57

2 Direct Methods **61**
2.1 Introduction 61
2.2 Direct Methods 63
2.2.1 Gaussian Elimination 63
2.2.2 The LU Decomposition 66
2.2.3 The LDU Decomposition 68
2.2.4 Inverse Iteration 68
2.3 Direct Methods and Markov Chains 70
2.3.1 Handling the Singularity 71
2.3.2 To Transpose or Not to Transpose 77
2.4 Implementation Considerations 78
2.4.1 Implementation in Two-Dimensional Storage Arrays 78
2.4.2 Compact Storage Schemes for Direct Methods 78
2.4.3 Simultaneous Row Generation and Reduction 81
2.4.4 Back-Substitution and Normalization 84
2.5 The GTH Advantage 84
2.6 Sample Experiments with Direct Methods 87
2.6.1 An Interactive Computer System Model 88
2.6.2 Two Coxian Queues: A Highly Structured Example 91
2.7 Stability, Conditioning, and Error Analysis 101
2.7.1 Stable Algorithms and Well-Conditioned Problems 101
2.7.2 Floating-Point Representation of Real Numbers 104
2.7.3 Backward Error Analysis 106
2.7.4 Error Analysis for Gaussian Elimination 109
2.7.5 Condition Numbers, Residuals, and the Group Inverse 117

3 Iterative Methods **121**
3.1 The Power Method 121
3.1.1 Introduction 121
3.1.2 Application to an Arbitrary Matrix, A 122
3.1.3 Application to a Stochastic Matrix, P 124
3.1.4 Comparison with Matrix Powering 125
3.2 Jacobi, Gauss–Seidel, SOR, and Symmetric SOR 125
3.2.1 The Nonhomogeneous Case 126

3.2.2 The Method of Jacobi . . . 127
3.2.3 The Method of Gauss–Seidel . . . 128
3.2.4 The SOR Method . . . 131
3.2.5 The Symmetric SOR Method: SSOR . . . 132
3.2.6 Examples . . . 133
3.3 Block Iterative Methods . . . 138
3.3.1 MATLAB Code and Examples . . . 141
3.4 Preconditioned Power Iterations . . . 143
3.4.1 Gauss–Seidel, SOR, and SSOR Preconditionings . . . 144
3.4.2 *ILU* Preconditioning . . . 144
3.4.3 Examples . . . 146
3.4.4 Summary . . . 150
3.5 Implementation Considerations . . . 151
3.5.1 Sparse Storage Schemes . . . 151
3.5.2 Choice of an Initial Iteration Vector . . . 155
3.5.3 Normalization of Successive Approximations . . . 156
3.5.4 Testing for Convergence . . . 156
3.5.5 Choosing a Relaxation Parameter for SOR . . . 159
3.5.6 The Effect of State Space Orderings on Convergence . . . 163
3.6 Convergence Properties . . . 169
3.6.1 Definitions . . . 169
3.6.2 Convergence Theorems . . . 170
3.6.3 Application to Markov Chains . . . 176

4 Projection Methods **177**
4.1 Introduction . . . 177
4.1.1 Preliminaries . . . 177
4.1.2 General Projection Processes . . . 179
4.1.3 Projection Processes for Linear Systems . . . 179
4.1.4 Projection Processes for Eigenvalue Problems . . . 181
4.1.5 Application to Markov Chains . . . 181
4.2 Simultaneous Iteration . . . 182
4.2.1 A Generic Subspace Iteration Algorithm . . . 182
4.2.2 "LOPSI": A Lopsided Simultaneous Iteration Algorithm . . . 184
4.3 Krylov Subspaces . . . 187
4.3.1 Gram–Schmidt Orthogonalization Procedures . . . 188
4.4 GMRES and the Method of Arnoldi . . . 190
4.4.1 Arnoldi for Eigensolutions . . . 190
4.4.2 FOM — The Full Orthogonalization Method . . . 192
4.4.3 GMRES — The Generalized Minimal Residual Method . . . 194
4.4.4 Application to Markov Chains . . . 196
4.4.5 Examples . . . 196
4.4.6 Iterative, Incomplete, and Preconditioned Methods . . . 197

4.4.7 Examples, continued . . . 199
4.4.8 Implementation . . . 201
4.4.9 The Complete Iterative GMRES Algorithm with Preconditioning . . 205
4.5 Lanczos and Conjugate Gradients . . . 207
4.5.1 The Symmetric Lanczos Algorithm . . . 207
4.5.2 The Unsymmetric Lanczos Algorithm . . . 210
4.5.3 The "Look-Ahead" Lanczos Algorithm . . . 214
4.5.4 CG – The Conjugate Gradient Algorithm . . . 215
4.5.5 CGNR — Conjugate Gradient for the Normal Equations . . . 218
4.5.6 BCG and CGS — Conjugate Gradient for Nonsymmetric Systems . 220
4.5.7 Preconditioning . . . 222
4.5.8 Examples . . . 224
4.6 MATLAB Programs . . . 225

5 Block Hessenberg Matrices and Solution by Recursion 231
5.1 Hessenberg Matrices . . . 231
5.1.1 Definitions . . . 231
5.1.2 Standard Queueing Recursions as Forward Substitutions . . . 232
5.1.3 Block Hessenberg Matrices . . . 233
5.2 Block Recursive Methods . . . 234
5.2.1 The Recursive Procedure . . . 235
5.2.2 Example: A Telephone System with N Lines and K Operators . . . 240
5.3 The Matrix-Geometric Approach . . . 258
5.3.1 Introduction . . . 258
5.3.2 Matrix Geometric Solutions: The Matrix R . . . 258
5.3.3 Implementation: Computing R and π_0 . . . 260
5.3.4 Example: The $\lambda/C_2/1$ Queue . . . 262
5.3.5 Alternative Methods for Finding R . . . 264
5.3.6 The Quasi-Birth-Death (QBD) Case . . . 266
5.4 Explicit Solution of Matrix-Geometric Problems . . . 270
5.4.1 Queueing Systems for Which R May Be Explicitly Computed . . . 272
5.4.2 Systems for Which R Must Be Computed Explicitly . . . 278
5.4.3 Systems for Which R Must Be Computed Iteratively . . . 280
5.4.4 Example: A Markovian Queue with N Servers Subject to Breakdowns and Repairs . . . 281

6 Decompositional Methods 285
6.1 NCD Markov Chains . . . 285
6.1.1 Introduction and Background . . . 285
6.1.2 Definitions . . . 286
6.1.3 Block Solutions . . . 287
6.1.4 The Coupling Matrix . . . 289
6.1.5 The NCD Approximation — A Rayleigh–Ritz Refinement Step . . . 292
6.2 Stochastic Complementation . . . 294

6.2.1 Definition . . . 294
6.2.2 Properties of the Stochastic Complement . . . 296
6.2.3 Computing Stationary Distributions by Stochastic Complementation 299
6.2.4 Relationship with Block Gaussian Elimination . . . 300
6.2.5 Computational Aspects of Stochastic Complementation . . . 302
6.3 Iterative Aggregation/Disaggregation Methods . . . 307
6.3.1 Introduction . . . 307
6.3.2 The Basic IAD Algorithm . . . 307
6.3.3 The Takahashi IAD Algorithm . . . 311
6.3.4 Other IAD Variants . . . 316
6.3.5 Restructuring an NCD Matrix . . . 316
6.3.6 Implementation Considerations . . . 320
6.3.7 MATLAB Experiments . . . 323
6.3.8 A Large Experiment . . . 325
6.4 Convergence Properties and Behavior . . . 332
6.4.1 Necessary Conditions for a "Regular" NCD Stochastic Matrix . . . 332
6.4.2 Three Lemmas to Characterize Approximations . . . 336
6.4.3 A Convergence Theorem . . . 340

7 *P*-Cyclic Markov Chains 343
7.1 Introduction . . . 343
7.2 Directed Graphs and *P*-Cyclic Matrices . . . 348
7.2.1 Graph Terminology and Definitions . . . 348
7.2.2 Primitive and Cyclic Matrices . . . 352
7.3 *p*-Cyclic Markov Chains . . . 355
7.3.1 The Embedded Markov Chain . . . 355
7.3.2 Markov Chains with Periodic Graphs . . . 355
7.3.3 Computation of the Periodicity . . . 356
7.4 Numerical Methods Applied to *p*-Cyclic Matrices . . . 359
7.4.1 Direct Methods . . . 359
7.4.2 The Gauss–Seidel Iterative Method . . . 360
7.4.3 Numerical Experiments . . . 361
7.5 Reduced Schemes . . . 364
7.5.1 Reduced Schemes Associated with a Stochastic Matrix . . . 364
7.5.2 Reduced Schemes Associated with an Infinitesimal Generator . . . 366
7.5.3 Numerical Methods Based on the Reduced Scheme . . . 367
7.5.4 Numerical Experiments . . . 369
7.6 IAD Methods for NCD, *p*-Cyclic Markov Chains . . . 371
7.6.1 Iterative Aggregation/Disaggregation (IAD) Methods . . . 372
7.6.2 Ordering for Periodicity and Decomposability . . . 372
7.6.3 Application to *p*-Cyclic Matrices . . . 373
7.7 Block SOR and Optimum Relaxation . . . 374
7.7.1 Introduction . . . 374

7.7.2 A 3-Cyclic Queueing Network Example . . . 377
7.7.3 A p-step Iteration Procedure . . . 380
7.7.4 Convergence Conditions for Block SOR . . . 391
7.7.5 Applicable Values for the Relaxation Parameter . . . 394
7.7.6 Convergence Testing in the Extended Sense . . . 398
7.7.7 Optimal Convergence Factors and Associated ω values . . . 400
7.7.8 Computing the Subvector Multipliers . . . 403

8 Transient Solutions **407**
8.1 Introduction . . . 407
8.2 Uniformization . . . 408
8.2.1 The Basic Concepts . . . 408
8.2.2 The Truncation Error . . . 410
8.2.3 Implementation . . . 411
8.3 Methods Applicable to Small State Spaces . . . 413
8.3.1 Matrix Decompositional Methods . . . 414
8.3.2 Matrix Scaling and Powering . . . 416
8.4 Ordinary Differential Equation (ODE) Solvers . . . 422
8.4.1 Ordinary Differential Equations . . . 423
8.4.2 Numerical Solutions and Stability . . . 426
8.4.3 Elementary Numerical Algorithms . . . 429
8.4.4 Stiff ODEs . . . 434
8.4.5 Single-Step Methods . . . 436
8.4.6 Multistep Methods . . . 444
8.4.7 Software Sources and Comparisons . . . 450
8.5 Krylov Subspace Methods . . . 453
8.5.1 Introduction . . . 453
8.5.2 The Basic Krylov Subspace Approach . . . 454
8.5.3 Corrected Schemes and Error Bounds . . . 457
8.5.4 Error Estimates and Step Size Control . . . 458
8.6 Selected Experiments . . . 460

9 Stochastic Automata Networks **463**
9.1 Introduction . . . 463
9.2 Noninteracting Stochastic Automata . . . 464
9.2.1 Independent, Two-Dimensional Markov Chains . . . 464
9.2.2 Basic Properties of Tensor Algebra . . . 464
9.2.3 Probability Distributions . . . 466
9.3 Interacting Stochastic Automata . . . 467
9.3.1 A Simple Example . . . 467
9.3.2 Functional Transition Rates and Synchronizing Events . . . 467
9.3.3 The Effect of Synchronizing Events . . . 469
9.3.4 The Effect of Functional Transition Rates . . . 472
9.4 Computing Probability Distributions . . . 474

9.5 Multiplication with a Vector 475
9.5.1 The Nonfunctional Case 476
9.5.2 Multiplication in the Presence of Functional Elements 479
9.5.3 The Use of Symmetric Permutations 480
9.5.4 When All Else Fails 481
9.6 Iterative Solution Methods 483
9.6.1 Unpreconditioned Methods 483
9.6.2 Preconditioning 484
9.7 A Queueing Network Example 486

10 Software **491**
10.1 The Categories of Available Software 491
10.1.1 Introduction 491
10.1.2 Libraries of Numerical Algorithms 492
10.1.3 Queueing Networks 493
10.1.4 Stochastic Petri Nets 496
10.1.5 Other Software 498
10.2 MARCA — MARkov Chain Analyzer 500
10.2.1 Basic Concepts and Terminology 500
10.2.2 Model Definition in MARCA 503
10.2.3 Numerical Methods 504
10.3 XMARCA: A Graphical Interface for Queueing Networks 505
10.3.1 Introduction 505
10.3.2 Building a Queueing Network Model 506
10.3.3 Converting the Graphical Representation 509
10.3.4 Matrix Generation and Numerical Solutions 509
10.3.5 Displaying Results 512

Bibliography **513**

Index **529**