CONTENTS

Preface v
Translator's Note vi
Contents vii

CHAPTER I

Fundamental Concepts and Theorems

1. Simple Homogeneous Markov Chains with a Finite Number of States and Discrete Time 1
2. Stochastic Matrices 4
3. Fundamental Properties of Non-Negative Matrices . . . 5
4. Fundamental Properties of Stochastic Matrices 7
5. Perron's Formula 15
6. Fundamental Formulae for Chains C_n 18
7. Some Consequences of the Fundamental Formulae for the Chains C_n 20
8. Essential and Inessential Classes of States of the System S and Decomposability and Indecomposability of the Matrix P 26
9. Subclasses of Essential States of the System S and Cyclic Matrices. 28
10. Classification of Chains C_n 32

CHAPTER II

Decomposable and Non-decomposable Acyclic Chains C_n

11. Decomposable and Non-decomposable Stochastic Matrices 34
12. Conditions for the Decomposability and Non-decomposability of the Matrix P 37
13. Regular Chains C_n 46
14. Supplementary Conditions for the Regularity of C_n . . . 53
15. Influence of the States of the System S on Each Other in a Regular Chain C_n. 55
16. Example: The Original Markov Chain 58
17. Transience and Permanence Matrices of a Chain C_n . . 61
18. An Independent Representation of Probabilities $p_{\alpha_\nu\beta_h}^{(\nu)}$ for a Decomposable Chain C_n. 70

CHAPTER III

Non-decomposable Cyclic Chains

19. Cyclic Stochastic Matrices 82
20. Fundamental Theorem on the Roots of a Non-decomposable Cyclic Matrix 86
21. Properties of the Minors of the Determinant $P(\lambda)$ of a Non-decomposable Cyclic Chain C_n 91
22. Powers of the Non-decomposable Cyclic Matrix P . . . 100
23. Transition and Absolute Probabilities of a Non-decomposable Cyclic Chain C_n. 102
24. Example. 108
25. Another Method for the Computation of the Probabilities $p_{\alpha\beta}^{(s)}$ for a Cyclic Chain C_n 113
26. Polycyclic Chains. 117
27. Bicyclic Chains. 121
28. Solution of the First Fundamental Problem for Bicyclic Chains of the First Kind 122

29. Properties of the Minors of the Determinant $P(\lambda)$ of a Bicyclic Chain of the First Kind 128
30. Solution of the Second and Third Problems for Bicyclic Chains of the First Kind 134
31. Bicyclic Chains of the Second Kind 140
32. General Remarks on Cyclic Processes 141
33. An Example of the Application of a Cyclic Chain with Absorbing States to Chemical Chain Reactions 148
34. The Fundamental Class of Polycyclic Chains 151

CHAPTER IV

Characteristic Functions of Discrete Markov Chains

35. Introductory Remarks 162
36. Characteristic Functions of Chains C_n 163
37. A New Form of the Characteristic Function φ_k 171
38. Moments of the Chains C_n 174
39. Investigation of Variances and Covariances of C_n 184
40. Investigation of Variances and Covariances of C_n (continuation) 196
41. Variances and Covariances of Decomposable Chains C_n . 199
42. On Limiting Distributions for Stochastic Processes with Discrete Time 206
43. The Limiting Distribution of Frequencies in a Chain C_n . 213
44. Mean Frequencies of States of the System S 220
45. Discrete Random Variables, Governed by a Simple Markov Chain 223
46. Investigation of the Distribution of the Sum of the Variables $u_0, u_1, \ldots, u_{s-1}$ in the Chain $C_n(X)$ 225
47. Investigation of the Variance of the Sum S_s, when the Chain $C_n(X)$ is Non-decomposable and Cyclic 231
48. Fréchet's Form for the Variance σ_x^2 in a Non-decomposable, Regular Chain $C_n(X)$ 241

CHAPTER V

Chain Correlations

49. Introductory Remarks. 244
50. The Simplest Case of a Chain Correlation Between Two Random Variables; the Chain $C_1(X, Y)$ and its Fundamental Properties. 245
51. Moments of the First and Second Order of the Chains $C_m(Y)$ and $C_1(X, Y)$ 254
52. Characteristic Functions of the Chains $C_m(Y)$ and $C_1(X, Y)$ and Limiting Theorems on the Distribution of the Sums of the Values of X and Y. 261
53. The Inversion of the Chain $C_1(X, Y)$ 266
54. The Chain $C_2(X, Y)$. 268
55. Markov's Chain Correlation 271
56. Other Forms of Chain Correlation 275

CHAPTER VI

Markov-Bruns Chains

57. Introduction 282
58. The Simplest Case of a Markov-Bruns Chain. 282
59. Generalization of Markov-Bruns Chains. 290
60. The General Case of Markov-Bruns Chains 297
61. Study of the Distribution of Frequencies of Separate Compound Events in a General Markov-Bruns Chain 304
62. Markov-Bruns Chains for Random Variables 312

CHAPTER VII

Complex Chains

63. Markov's Case 315
64. The General Case of a Doubly-Associated Chain 319
65. Characteristic Functions of the Chain $C_n^{(2)}$ 323
66. Multiply-Associated Chains 325
67. Infinitely Complex Chains 328
68. Stationary Chains C_n^* 335

CHAPTER VIII

Supplements and Applications

69. The Law of Large Numbers for the Chains C_n and $C_n(X)$ 338
70. The Law of the Iterated Logarithm for the Chain $C_n(X)$. 340
71. Reverse Chains. 349
72. Reverse Cyclic Chains. 354
73. Chains C_n with No Initial Time Point. 357
74. Probabilities of Repetitions of Cycles in Polycyclic Chains 360
75. Statistical Problems Associated with the Chains C_n . . . 366
76. The Coefficient of Rigidity of a Chain. 375
77. Applications of Markov and Markov-Bruns Chains to the Study of Randomness. 380
78. Application of Markov Chains to Geophysical Problems. 391
79. Non-homogeneous Markov Chains 397

Bibliography. 406
More Recent Bibliography for Finite Discrete Markov Chains 409