

# Contents

|          |  |          |
|----------|--|----------|
| <b>1</b> | <b>Rank one perturbations</b>  | <b>9</b> |
| 1.1      | Bounded perturbations . . . . .  | 9        |
| 1.1.1    | Resolvent analysis . . . . .   | 9        |
| 1.1.2    | Infinite coupling . . . . .  | 14       |
| 1.2      | Krein's formula . . . . .  | 15       |
| 1.2.1    | Bounded and singular perturbations . . . . .   | 15       |
| 1.2.2    | Scale of Hilbert spaces . . . . .  | 17       |
| 1.2.3    | Form bounded and form unbounded<br>perturbations . . . . .                           | 19       |
| 1.2.4    | Rank one perturbations and the extension<br>theory for symmetric operators . . . . . | 21       |
| 1.2.5    | The extension theory and Krein's formula . . . . .                                   | 23       |
| 1.2.6    | $Q$ -function for rank one perturbations . . . . .                                   | 28       |
| 1.3      | Singular rank one perturbations . . . . .  | 30       |
| 1.3.1    | Form bounded rank one singular perturbations . . . . .                               | 30       |
| 1.3.2    | Family of rank one form unbounded<br>perturbations . . . . .                         | 32       |
| 1.3.3    | Singular rank one form unbounded perturbations of<br>homogeneous operators . . . . . | 35       |
| 1.3.4    | Resolvent formulas . . . . .   | 38       |
| 1.4      | Approximations of singular rank one perturbations . . . . .                          | 41       |
| 1.4.1    | Norm convergence of the approximations . . . . .                                     | 41       |
| 1.4.2    | Strong resolvent convergence of the<br>approximations . . . . .                      | 46       |
| 1.5      | Differential operators with rank one singular perturbations . . . . .                | 49       |
| 1.5.1    | Point interactions in dimension three . . . . .                                      | 49       |
| 1.5.2    | Perturbations of the first derivative operator . . . . .                             | 58       |
| 1.5.3    | Dirac operator with a pseudopotential . . . . .                                      | 60       |

|          |   |            |
|----------|---|------------|
| <b>2</b> | <b>Generalized rank one perturbations</b>   | <b>63</b>  |
| 2.1      | Krein's formula for the generalized resolvents . . . . .                                | 63         |
| 2.1.1    | Generalized self-adjoint extensions and<br>generalized resolvents . . . . .             | 63         |
| 2.1.2    | Generalized rank one perturbations . . . . .  | 64         |
| 2.2      | Model generalized perturbations . . . . .   | 69         |
| 2.2.1    | Introduction . . . . .  | 69         |
| 2.2.2    | Model perturbations I: densely defined restricted oper-<br>ators . . . . .              | 70         |
| 2.2.3    | Model perturbations II: non-densely defined<br>construction . . . . .                   | 81         |
| 2.2.4    | Generalized resolvents and model generalized<br>perturbations . . . . .                 | 90         |
| 2.3      | Generalized point interactions . . . . .  | 92         |
| 2.3.1    | Generalized point interaction in dimension three . . . . .                              | 92         |
| 2.3.2    | Generalized delta interaction in dimension one . . . . .                                | 99         |
| <b>3</b> | <b>Finite rank perturbations and distribution theory</b>                                | <b>111</b> |
| 3.1      | Finite rank perturbations . . . . .   | 111        |
| 3.1.1    | Preliminaries . . . . .   | 111        |
| 3.1.2    | Form bounded finite rank perturbations . . . . .  | 116        |
| 3.1.3    | Form unbounded finite rank perturbations . . . . .                                      | 120        |
| 3.1.4    | Generalized finite rank perturbations . . . . .   | 125        |
| 3.2      | Point interactions for differential operators and distribution<br>theory . . . . .      | 130        |
| 3.2.1    | Point interactions for differential operators as<br>finite rank perturbations . . . . . | 130        |
| 3.2.2    | Distribution theory for discontinuous test functions . . . . .                          | 134        |
| 3.2.3    | Differential operator of order $n$ in one<br>dimension . . . . .                        | 140        |
| 3.2.4    | Second order differential operator in one<br>dimension . . . . .                        | 142        |
| <b>4</b> | <b>Scattering theory for finite rank perturbations</b>                                  | <b>159</b> |
| 4.1      | Scattering theory for rank one perturbations . . . . .                                  | 159        |
| 4.1.1    | Rank one perturbations and operators with a simple<br>spectrum . . . . .                | 160        |
| 4.1.2    | Invariance of the absolutely continuous<br>spectrum . . . . .                           | 162        |
| 4.1.3    | Wave operators and scattering operator<br>for rank one perturbations . . . . .          | 169        |

|          |  |            |
|----------|--|------------|
| 4.2      | Scattering for self-adjoint extensions . . . . .   | 175        |
| 4.2.1    | Wave operators for self-adjoint extensions . . . . .   | 175        |
| 4.2.2    | Scattering operator for self-adjoint extensions . . . . .  | 177        |
| 4.2.3    | Scattering matrix for self-adjoint extensions . . . . .  | 184        |
| 4.3      | Scattering theory for finite rank perturbations . . . . .  | 188        |
| 4.3.1    | Scattering theory for finite rank perturbations . . . . .  | 188        |
| 4.3.2    | Scattering matrix for rank two perturbations . . . . .   | 190        |
| 4.3.3    | Scattering matrix for generalized perturbations . . . . .  | 191        |
| <b>5</b> | <b>Krein's formula for infinite deficiency indices and<br/>two-body problems</b>                                 | <b>195</b> |
| 5.1      | Infinite rank perturbations . . . . .  | 195        |
| 5.1.1    | Krein's formula for infinite deficiency indices . . . . .  | 195        |
| 5.1.2    | Generalized perturbations of infinite rank . . . . .   | 198        |
| 5.1.3    | Resolvent formula for functionals . . . . .  | 202        |
| 5.2      | Two-body problems . . . . .  | 205        |
| 5.2.1    | Two-body operator with interaction of rank one . . . . .   | 205        |
| 5.2.2    | Two-body operator with generalized interaction of rank<br>one . . . . .  | 219        |
| <b>6</b> | <b>Few-body problems</b>   | <b>227</b> |
| 6.1      | Few-body formula I: self-adjoint extensions . . . . .  | 227        |
| 6.1.1    | Tensor structure of the few-body Hilbert space . . . . .   | 227        |
| 6.1.2    | Few-body operator with $\mathcal{H}_{-1}$ interaction . . . . .  | 232        |
| 6.1.3    | Few-body operator with $\mathcal{H}_{-2}$ interaction . . . . .  | 238        |
| 6.1.4    | Self-adjointness of the few-body operator<br>with strongly separable interactions . . . . .                      | 244        |
| 6.1.5    | Few-body operator with delta interaction. . . . .  | 247        |
| 6.2      | Few-body formula II: generalized self-adjoint extensions . . . . .   | 248        |
| 6.2.1    | Generalized unperturbed operator . . . . .   | 248        |
| 6.2.2    | Inner-cluster generalized interaction . . . . .  | 250        |
| 6.2.3    | Few-body operator with generalized interaction . . . . .   | 256        |
| 6.2.4    | Self-adjointness of the few-body operator with infinites-<br>imally separable generalized interactions . . . . . | 265        |
| <b>7</b> | <b>Three-body models in one dimension</b>  | <b>273</b> |
| 7.1      | Schrödinger operators constructed using self-adjoint extensions  | 273        |
| 7.1.1    | Tensor structure of the Hilbert space . . . . .  | 273        |
| 7.1.2    | Definition of the Schrödinger operator as<br>a self-adjoint operator. . . . .                                    | 274        |
| 7.1.3    | Spectrum, eigenfunctions and Bethe Ansatz . . . . .  | 277        |

|          |   |            |
|----------|---|------------|
| 7.2      | Operators with generalized delta interactions . . . . .   | 281        |
| 7.2.1    | Two-body generalized delta interactions . . . . .   | 281        |
| 7.2.2    | Three-body Schrödinger operator with two-body generalized delta interactions . . . . .            | 283        |
| 7.2.3    | Symmetry group . . . . .  | 285        |
| 7.2.4    | Outgoing wave and Sommerfeld–Maluzhinetz transformation . . . . .                                 | 293        |
| 7.2.5    | Solution of the functional equations . . . . .  | 298        |
| 7.2.6    | Properties of the outgoing wave . . . . .   | 305        |
| 7.2.7    | Spectrum and scattering matrix . . . . .  | 315        |
| 7.2.8    | Calculation of the scattering matrix . . . . .  | 318        |
| <b>A</b> | <b>Historical remarks.</b>  | <b>327</b> |
| A.1      | Extension theory for symmetric operators . . . . .  | 327        |
| A.1.1    | Extension theory and Nevanlinna $R$ -functions . . . . .  | 327        |
| A.1.2    | Symplectic structure and von Neumann construction . . . . .                                       | 330        |
| A.2      | Finite rank perturbations . . . . .   | 332        |
| A.2.1    | Definition of singular finite rank perturbations . . . . .  | 332        |
| A.2.2    | Generalized singular perturbations . . . . .  | 334        |
| A.2.3    | Singular perturbations defined by forms . . . . .   | 336        |
| A.3      | Point interactions . . . . .  | 337        |
| A.3.1    | Definition of the point interactions in $\mathbf{R}^1, \mathbf{R}^2$ and $\mathbf{R}^3$ . . . . . | 337        |
| A.3.2    | Approximations of point interactions . . . . .  | 340        |
| A.3.3    | Point interactions and inverse problems . . . . .   | 341        |
| A.4      | Few-body problems . . . . .   | 341        |
| A.4.1    | Three-body problems in $\mathbf{R}^3$ and $\mathbf{R}^2$ . . . . .                                | 341        |
| A.4.2    | Few-body problems in $\mathbf{R}^1$ . . . . .   | 344        |
| A.5      | Recent developments . . . . .   | 346        |
| A.5.1    | Sphere interactions . . . . .   | 346        |
| A.5.2    | Interactions on low dimensional manifolds . . . . .   | 346        |
| A.5.3    | Relativistic point interactions . . . . .   | 347        |
| A.5.4    | Self-adjoint operators on graphs . . . . .  | 348        |
| A.5.5    | Acoustic problems . . . . .   | 349        |
| A.5.6    | Magnetic field, Aharonov–Bohm effect and anyons . . . . .   | 349        |
| A.5.7    | Time dependent interactions . . . . .   | 350        |
| A.5.8    | Solid state . . . . .   | 350        |
| A.5.9    | Further developments . . . . .  | 351        |
|          | <b>Bibliography</b>   | <b>353</b> |
|          | <b>Index</b>  | <b>427</b> |