Contents

Introduction to the Subject 1
General Basic Ideas 4

CHAPTER 37
Introductory Typical Examples 12
§37.1. Real Functions in $\mathbb{R}^1$ 13
§37.2. Convex Functions in $\mathbb{R}^1$ 15
§37.3. Real Functions in $\mathbb{R}^N$, Lagrange Multipliers, Saddle Points, and Critical Points 16
§37.4. One-Dimensional Classical Variational Problems and Ordinary Differential Equations, Legendre Transformations, the Hamilton–Jacobi Differential Equation, and the Classical Maximum Principle 20
§37.5. Multidimensional Classical Variational Problems and Elliptic Partial Differential Equations 41
§37.6. Eigenvalue Problems for Elliptic Differential Equations and Lagrange Multipliers 43
§37.7. Differential Inequalities and Variational Inequalities 44
§37.8. Game Theory and Saddle Points, Nash Equilibrium Points and Pareto Optimization 47
§37.9. Duality between the Methods of Ritz and Trefftz, Two-Sided Error Estimates 50
§37.10. Linear Optimization in $\mathbb{R}^N$, Lagrange Multipliers, and Duality 51
§37.11. Convex Optimization and Kuhn–Tucker Theory 55
§37.12. Approximation Theory, the Least-Squares Method, Deterministic and Stochastic Compensation Analysis 58
§37.13. Approximation Theory and Control Problems 64

§37.14. Pseudoinverses, Ill-Posed Problems and Tihonov Regularization 65
§37.15. Parameter Identification 71
§37.16. Chebyshev Approximation and Rational Approximation 73
§37.17. Linear Optimization in Infinite-Dimensional Spaces, Chebyshev Approximation, and Approximate Solutions for Partial Differential Equations 76
§37.18. Splines and Finite Elements 79
§37.19. Optimal Quadrature Formulas 80
§37.20. Control Problems, Dynamic Optimization, and the Bellman Optimization Principle 84
§37.21. Control Problems, the Pontrjagin Maximum Principle, and the Bang-Bang Principle 89
§37.22. The Synthesis Problem for Optimal Control 92
§37.23. Elementary Provable Special Case of the Pontrjagin Maximum Principle 93
§37.24. Control with the Aid of Partial Differential Equations 96
§37.25. Extremal Problems with Stochastic Influences 97
§37.26. The Courant Maximum-Minimum Principle. Eigenvalues, Critical Points, and the Basic Ideas of the Ljusternik–Schnirelman Theory 102
§37.27. Critical Points and the Basic Ideas of the Morse Theory 105
§37.28. Singularities and Catastrophe Theory 115
§37.29. Basic Ideas for the Construction of Approximate Methods for Extremal Problems 132

TWO FUNDAMENTAL EXISTENCE AND UNIQUENESS PRINCIPLES

CHAPTER 38
Compactness and Extremal Principles 145
§38.1. Weak Convergence and Weak* Convergence 147
§38.2. Sequential Lower Semicontinuous and Lower Semicontinuous Functionals 149
§38.3. Main Theorem for Extremal Problems 151
§38.4. Strict Convexity and Uniqueness 152
§38.5. Variants of the Main Theorem 153
§38.6. Application to Quadratic Variational Problems 155
§38.7. Application to Linear Optimization and the Role of Extreme Points 157
§38.8. Quasisolutions of Minimum Problems 158
§38.9. Application to a Fixed-Point Theorem 161
§38.10. The Palais–Smale Condition and a General Minimum Principle 161
§38.11. The Abstract Entropy Principle 163

CHAPTER 39
Convexity and Extremal Principles 168
§39.1. The Fundamental Principle of Geometric Functional Analysis 170
§39.2. Duality and the Role of Extreme Points in Linear Approximation Theory 172
§39.3. Interpolation Property of Subspaces and Uniqueness 175
§39.4. Ascent Method and the Abstract Alternation Theorem 177
§39.5. Application to Chebyshev Approximation 180

EXTREMAL PROBLEMS WITHOUT SIDE CONDITIONS

CHAPTER 40
Free Local Extrema of Differentiable Functionals and the Calculus of Variations 189
§40.1. nth Variations, G-Derivative, and F-Derivative 191
§40.2. Necessary and Sufficient Conditions for Free Local Extrema 193
§40.3. Sufficient Conditions by Means of Comparison Functionals and Abstract Field Theory 195
§40.4. Application to Real Functions in $\mathbf{R}^N$ 195
§40.5. Application to Classical Multidimensional Variational Problems in Spaces of Continuously Differentiable Functions 196
§40.6. Accessory Quadratic Variational Problems and Sufficient Eigenvalue Criteria for Local Extrema 200
§40.7. Application to Necessary and Sufficient Conditions for Local Extrema for Classical One-Dimensional Variational Problems 203

CHAPTER 41
Potential Operators 229
§41.1. Minimal Sequences 232
§41.2. Solution of Operator Equations by Solving Extremal Problems 233
§41.3. Criteria for Potential Operators 234
§41.4. Criteria for the Weak Sequential Lower Semicontinuity of Functionals 235
§41.5. Application to Abstract Hammerstein Equations with Symmetric Kernel Operators 237
§41.6. Application to Hammerstein Integral Equations 239

CHAPTER 42
Free Minima for Convex Functionals, Ritz Method and the Gradient Method 244
§42.1. Convex Functionals and Convex Sets 245
§42.2. Real Convex Functions 246

§42.3. Convexity of F, Monotonicity of F', and the Definiteness of the Second Variation 247
§42.4. Monotone Potential Operators 249
§42.5. Free Convex Minimum Problems and the Ritz Method 250
§42.6. Free Convex Minimum Problems and the Gradient Method 252
§42.7. Application to Variational Problems and Quasilinear Elliptic Differential Equations in Sobolev Spaces 255

EXTREMAL PROBLEMS WITH SMOOTH SIDE CONDITIONS

CHAPTER 43
Lagrange Multipliers and Eigenvalue Problems 273
§43.1. The Abstract Basic Idea of Lagrange Multipliers 274
§43.2. Local Extrema with Side Conditions 276
§43.3. Existence of an Eigenvector Via a Minimum Problem 278
§43.4. Existence of a Bifurcation Point Via a Maximum Problem 279
§43.5. The Galerkin Method for Eigenvalue Problems 281
§43.6. The Generalized Implicit Function Theorem and Manifolds in B-Spaces 282
§43.7. Proof of Theorem 43.C 288
§43.8. Lagrange Multipliers 289
§43.9. Critical Points and Lagrange Multipliers 291
§43.10. Application to Real Functions in $\mathbb{R}^N$ 293
§43.11. Application to Information Theory 294
§43.12. Application to Statistical Physics. Temperature as a Lagrange Multiplier 296
§43.13. Application to Variational Problems with Integral Side Conditions 299
§43.14. Application to Variational Problems with Differential Equations as Side Conditions 300

CHAPTER 44
Ljusternik–Schnirelman Theory and the Existence of Several Eigenvectors 313
§44.1. The Courant Maximum-Minimum Principle 314
§44.2. The Weak and the Strong Ljusternik Maximum-Minimum Principle for the Construction of Critical Points 316
§44.3. The Genus of Symmetric Sets 319
§44.4. The Palais–Smale Condition 321
§44.5. The Main Theorem for Eigenvalue Problems in Infinite-Dimensional B-spaces 324
§44.6. A Typical Example 328
§44.7. Proof of the Main Theorem 330

§44.8. The Main Theorem for Eigenvalue Problems in Finite-Dimensional B-Spaces 335
§44.9. Application to Eigenvalue Problems for Quasilinear Elliptic Differential Equations 336
§44.10. Application to Eigenvalue Problems for Abstract Hammerstein Equations with Symmetric Kernel Operators 337
§44.11. Application to Hammerstein Integral Equations 339
§44.12. The Mountain Pass Theorem 339

CHAPTER 45
Bifurcation for Potential Operators 351
§45.1. Krasnoselskii's Theorem 351
§45.2. The Main Theorem 352
§45.3. Proof of the Main Theorem 354

EXTREMAL PROBLEMS WITH GENERAL SIDE CONDITIONS

CHAPTER 46
Differentiable Functionals on Convex Sets 363
§46.1. Variational Inequalities as Necessary and Sufficient Extremal Conditions 363
§46.2. Quadratic Variational Problems on Convex Sets and Variational Inequalities 364
§46.3. Application to Partial Differential Inequalities 365
§46.4. Projections on Convex Sets 366
§46.5. The Ritz Method 367
§46.6. The Projected Gradient Method 368
§46.7. The Penalty Functional Method 370
§46.8. Regularization of Linear Problems 372
§46.9. Regularization of Nonlinear Problems 375

CHAPTER 47
Convex Functionals on Convex Sets and Convex Analysis 379
§47.1. The Epigraph 380
§47.2. Continuity of Convex Functionals 383
§47.3. Subgradient and Subdifferential 385
§47.4. Subgradient and the Extremal Principle 386
§47.5. Subgradient and the G-Derivative 387
§47.6. Existence Theorem for Subgradients 387
§47.7. The Sum Rule 388

§47.8. The Main Theorem of Convex Optimization 390
§47.9. The Main Theorem of Convex Approximation Theory 392
§47.10. Generalized Kuhn–Tucker Theory 392
§47.11. Maximal Monotonicity, Cyclic Monotonicity, and Subgradients 396
§47.12. Application to the Duality Mapping 399

CHAPTER 48
General Lagrange Multipliers (Dubovickii–Miljutin Theory) 407
§48.1. Cone and Dual Cone 408
§48.2. The Dubovickii–Miljutin Lemma 411
§48.3. The Main Theorem on Necessary and Sufficient Extremal Conditions for General Side Conditions 413
§48.4. Application to Minimum Problems with Side Conditions in the Form of Equalities and Inequalities 416
§48.5. Proof of Theorem 48.B 419
§48.6. Application to Control Problems (Pontrjagin's Maximum Principle) 422
§48.7. Proof of the Pontrjagin Maximum Principle 426
§48.8. The Maximum Principle and Classical Calculus of Variations 433
§48.9. Modifications of the Maximum Principle 435
§48.10. Return of a Spaceship to Earth 437

SADDLE POINTS AND DUALITY

CHAPTER 49
General Duality Principle by Means of Lagrange Functions and Their Saddle Points 457
§49.1. Existence of Saddle Points 457
§49.2. Main Theorem of Duality Theory 460
§49.3. Application to Linear Optimization Problems in B-Spaces 463

CHAPTER 50
Duality and the Generalized Kuhn–Tucker Theory 479
§50.1. Side Conditions in Operator Form 479
§50.2. Side Conditions in the Form of Inequalities 482

CHAPTER 51
Duality, Conjugate Functionals, Monotone Operators and Elliptic Differential Equations 487
§51.1. Conjugate Functionals 489
§51.2. Functionals Conjugate to Differentiable Convex Functionals 492

§51.3. Properties of Conjugate Functionals 493
§51.4. Conjugate Functionals and the Lagrange Function 496
§51.5. Monotone Potential Operators and Duality 499
§51.6. Applications to Linear Elliptic Differential Equations, Trefftz's Duality 502
§51.7. Application to Quasilinear Elliptic Differential Equations 506

CHAPTER 52
General Duality Principle by Means of Perturbed Problems and Conjugate Functionals 512
§52.1. The S-Functional, Stability, and Duality 513
§52.2. Proof of Theorem 52.A 515
§52.3. Duality Propositions of Fenchel–Rockafellar Type 517
§52.4. Application to Linear Optimization Problems in Locally Convex Spaces 519
§52.5. The Bellman Differential Inequality and Duality for Nonconvex Control Problems 521
§52.6. Application to a Generalized Problem of Geometrical Optics 525

CHAPTER 53
Conjugate Functionals and Orlicz Spaces 538
§53.1. Young Functions 538
§53.2. Orlicz Spaces and Their Properties 539
§53.3. Linear Integral Operators in Orlicz Spaces 541
§53.4. The Nemyckii Operator in Orlicz Spaces 542
§53.5. Application to Hammerstein Integral Equations with Strong Nonlinearities 542
§53.6. Sobolev–Orlicz Spaces 544

VARIATIONAL INEQUALITIES

CHAPTER 54
Elliptic Variational Inequalities 551
§54.1. The Main Theorem 551
§54.2. Application to Coercive Quadratic Variational Inequalities 552
§54.3. Semicoercive Variational Inequalities 553
§54.4. Variational Inequalities and Control Problems 556
§54.5. Application to Bilinear Forms 558
§54.6. Application to Control Problems with Elliptic Differential Equations 559
§54.7. Semigroups and Control of Evolution Equations 560

§54.8. Application to the Synthesis Problem for Linear Regulators 561
§54.9. Application to Control Problems with Parabolic Differential Equations 562

CHAPTER 55
Evolution Variational Inequalities of First Order in H-Spaces 568
§55.1. The Resolvent of Maximal Monotone Operators 569
§55.2. The Nonlinear Yosida Approximation 570
§55.3. The Main Theorem for Inhomogeneous Problems 570
§55.4. Application to Quadratic Evolution Variational Inequalities of First Order 572

CHAPTER 56
Evolution Variational Inequalities of Second Order in H-Spaces 577
§56.1. The Main Theorem 577
§56.2. Application to Quadratic Evolution Variational Inequalities of Second Order 578

CHAPTER 57
Accretive Operators and Multivalued First-Order Evolution Equations in B-Spaces 581
§57.1. Generalized Inner Products on B-Spaces 582
§57.2. Accretive Operators 583
§57.3. The Main Theorem for Inhomogeneous Problems with m-Accretive Operators 584
§57.4. Proof of the Main Theorem 585
§57.5. Application to Nonexpansive Semigroups in B-Spaces 593
§57.6. Application to Partial Differential Equations 594

Appendix 599

References 606

List of Symbols 637

List of Theorems 643

List of the Most Important Definitions 647

Index 651