## **Contents**

#### CHAPTER 1. PROBABILITY

1. Borel's Normal Number Theorem, 1

The Unit Interval—The Weak Law of Large Numbers—The Strong Law of Large Numbers—Strong Law versus Weak—Extending the Probabilities

2. Probability Measures, 15

Spaces—Classes of Sets—Probability Measures—Lebesgue Measure on the Unit Interval—Constructing σ-Fields\*

3. Existence and Extension, 29

Construction of the Extension—Uniqueness and the  $\pi$ - $\lambda$ Theorem—Monotone Classes—Completeness—Lebesgue Measure on the Unit Interval—Nonmeasurable Sets

4. Denumerable Probabilities, 40

General Formulas—Limit Sets—Independent Events—Subfields— The Borel-Cantelli Lemmas—The Zero-One Law—Strong Laws versus Weak

5. Simple Random Variables, 55

Definition—Independence—Existence of Independent Sequences— Expected Value—Inequalities

1

<sup>\*</sup> Asterisk indicates topics that may be omitted on a first reading.

6. The Law of Large Numbers, 70

The Strong Law—The Weak Law—Bernstein's Theorem—A Refinement of the Second Borel-Cantelli Lemma

7. Gambling Systems, 77

Gambler's Ruin—The Difference Equation—Selection Systems—Gambling Policies—Bold Play\*—Timid Play\*

8. Markov Chains, 93

Definitions—Higher-Order Transitions—An Existence Theorem— Transience and Persistence—Another Criterion for Persistence— Stationary Distributions—Exponential Convergence\*—Optimal Stopping\*

9. Large Deviations and the Law of the Iterated Logarithm,\* 119

Moment Generating Functions—Large Deviations—Chernoff's
Theorem—The Law of the Iterated Logarithm

#### CHAPTER 2. MEASURE

131

10. General Measures, 131

Classes of Sets—Conventions Involving ∞—Measures—Uniqueness

11. Outer Measure, 136

Outer Measure—Extension—An Approximation Theorem— Carathéodory's Condition\*

12. Measures in Euclidean Space, 143

Lebesgue Measure—Regularity—Specifying Measures on the Line—Specifying Measures in  $\mathbb{R}^k$ 

13. Measurable Functions and Mappings, 153

Measurable Mappings—Mappings into R<sup>k</sup>—Limits and Measurability—Transformations of Measures

14. Distribution Functions, 158

Distribution Functions—Exponential Distributions—Weak Convergence—Convergence of Types\*—Extremal Distributions\*—Cauchy's Equation

#### **CHAPTER 3. INTEGRATION**

15. The Integral, 171

Definition—Nonnegative Functions—Uniqueness

16. Properties of the Integral, 177

Equalities and Inequalities—Integration to the Limit—Densities—Change of Variable—Uniform Integrability—Complex Functions

17. Integral with Respect to Lebesgue Measure, 188

The Lebesgue Integral on the Line—The Riemann Integral—The Fundamental Theorem of Calculus—Change of Variable—The Lebesgue Integral in R<sup>k</sup>—Stieltjes Integrals

18. Product Measure and Fubini's Theorem, 196

Product Spaces—Product Measure—Fubini's Theorem— Integration by Parts—Products of Higher Order

19. Hausdorff Measure,\* 208

The Definition—The Normalizing Constant—Change of Variable—Calculations

# CHAPTER 4. RANDOM VARIABLES AND EXPECTED VALUES

219

20. Random Variables and Distributions, 219

Random Variables and Vectors—Subfields—Distributions— Independence—Sequences of Random Variables—Convolution— Convergence in Probability—The Glivenko-Cantelli Theorem\*

21. Expected Values, 237

Expected Value as Integral—Expected Values and Distributions—
Moments—Inequalities—Joint Integrals—Independence and
Expected Value—Moment Generating Functions

22. Sums of Independent Random Variables, 247

Kolmogorov's Zero-One Law—Kolmogorov's Inequality— Convergence of Random Series—The Strong Law of Large Numbers—The Weak Law and Moment Generating Functions— Random Taylor Series\*

### 23. The Poisson Process, 258

Characterization of the Exponential Distribution—The Poisson Process—Two Other Characterizations of the Poisson Process— Stochastic Processes

## 24. Queues and Random Walk,\* 268

The Single-Server Queue—Random Walk and Ladder Indices— Exponential Right Tail—Exponential Left Tail—Queue Size

# CHAPTER 5. CONVERGENCE OF DISTRIBUTIONS

280

## 25. Weak Convergence, 280

Definitions—Uniform Distribution Modulo 1\*—Convergence in Distribution—Convergence in Probability—Fundamental Theorems—Helley's Theorem—Integration to the Limit—The Diagonal Method

# 26. Characteristic Functions, 296

Definition—Moments and Derivatives—Independence—Inversion and the Uniqueness Theorem—The Continuity Theorem

# 27. The Central Limit Theorem, 308

Identically Distributed Summands—The Lindeberg and Lyapounov Theorems—Feller's Theorem\*—Dependent Variables\*

# 28. Infinitely Divisible Distributions,\* 322

Vague Convergence—The Possible Limits—Characterizing the Limit

# 29. Limit Theorems in $\mathbb{R}^k$ , 329

The Basic Theorems—Characteristic Functions—Normal Distributions in  $\mathbb{R}^k$ —The Central Limit Theorem—Skorohod's Theorem in  $\mathbb{R}^{k*}$ 

# 30. The Method of Moments,\* 342

The Moment Problem—Moment Generating Functions—Central Limit Theorem by Moments—Application to Sampling Theory—Application to Number Theory

31. Derivatives on the Line,\* 354

The Fundamental Theorem of Calculus—Derivatives of Integrals— Singular Functions—Integrals of Derivatives—Functions of Bounded Variation

32. The Radon-Nikodym Theorem, 372

Additive Set Functions—The Hahn Decomposition—Absolute Continuity and Singularity—The Main Theorem

33. Conditional Probability, 378

The Discrete Case—The General Case—Properties of Conditional Probability—Difficulties and Curiosities—Conditional Probability Distributions

34. Conditional Expectation, 395

Definition—Properties of Conditional Expectation—Conditional Distributions and Expectations—Sufficient Subfields\*—Minimum-Variance Estimation\*

35. Martingales, 407

Definition—Submartingales—Gambling—Functions of
Martingales—Inequalities—Convergence Theorems—Reversed
Martingales—Applications: Derivatives—Likelihood Ratios—
Bayes Estimation—The Zero-One Law—The Strong Law of Large
Numbers\*—The Hewitt-Savage Theorem\*—Exchangeable
Random Variables\*

#### CHAPTER 7. STOCHASTIC PROCESSES

430

36. Kolmogorov's Existence Theorem, 430

Stochastic Processes—Finite-Dimensional Distributions—Product Spaces—Kolmogorov's Existence Theorem—The Inadequacy of  $\mathcal{R}^T$ 

37. Brownian Motion, 442

Definition—Continuity of Paths—Measurable Processes— Irregularity of Brownian Motion Paths—The Strong Markov Property—Skorohod Embedding\*—Invariance\*

X	1	v

#### **CONTENTS**

38.	Sep	ara	bili	tv.*	467
	~~	u.u.	~	-,,,	707

Introduction—Definitions—Existence Theorems—Consequences	of
Separability—Separability in Product Space	•

NOTES ON THE PROBLEMS	478
BIBLIOGRAPHY	507
	501
LIST OF SYMBOLS	500
DIGIT OF STANDOLS	509
INDEX	
INDEX	511