CONTENTS

COMPLEMENTS TO CHAPTER IV: Some useful properties of jumps of cadlag processes.	xiii
CHAPTER V. GENERALITIES AND THE DISCRETE CASE	1
1. <u>Definitions and general properties</u> . Definition of martingales and supermartingales (1-2). Immediate properties (3-5). Examples (6). Right and left closed martingale and supermartingales (7).	1 s
2. <u>Doob's stopping theorem</u> . Transform of a martingale (8). The stopping theorem (finite case and comments (9-13). Extension to unbounded stopping times (14-1)	
3. Fundamental inequalities. Notation (19). The "maximal lemma" (20-23). Domination in L^p , $p > 1$ (24-25). Upcrossings and downcrossings (26-27).	13
4. Convergence and decomposition theorems. A.s. convergence of supermartingales (28-30) and martingales (31-33). Riesz (34-37) and Krickeberg (38) decompositions. Martingales over certain σ -finite spaces (39-43). A remark on convergence in L ¹ (44).	21
5. Some applications of the convergence theorems. The "Hunt lemma" (45) and "Lévy's Borel-Cantelli lemma" (46). Symmetric random variables (47-52). A theorem of Choquet and Deny (53-55). Applications to measure theory: the Radon-Nikodym	38

viii CONTENTS

theorem (56-57), existence of densities depending measurably on a parameter (58), the lifting theorem (59-60). A theorem of Rota (61-65) and its extension to pseudo-kernels (66-67). The Calderon-Zygmund lemma (68).

CHAPTER VI. CONTINUOUS PARAMETER MARTINGALES

65

1. Right continuous supermartingales.

66

Revision of discrete inequalities (1). Right and left limits with respect to a countable set (2-3). Existence of cadlag modifications (4). Regularization without completion (5). Right continuous supermartingales: extensions of the discrete case (6-7); Riesz decompositions (8-9); the stopping theorem (10-13). Properties peculiar to the continuous case: predictable form of the stopping theorem (14-16); zeros of positive supermartingales (17); increasing sequences of supermartingales (18-19). Processes bounded in L^1 or of class (D) (20). Properties of processes bounded in L^1 or of class (D) (20). Properties of processes bounded in L^1 (21-22). Criteria for belonging to class (D) (23-26). Local martingales (27-29); technical results (30-32). Krickeberg decomposition (33-36bis). Quasimartingales (37-39). Rao decomposition (40-42).

2. Projections and dual projections.

103

Definition and construction of projections (43-45). Comparison between ${}^{O}X$ and ${}^{D}X$ (46). Regularity of the paths of a projection (47-50). Increasing processes (51); structure (52-53); the case of processes indexed by $[0, \infty]$ (53 (c)). Calculating Stieltjes integrals (54-55). Change of time (56). Application to increasing processes (57-58). Characterization of optional or predictable increasing processes (59-60). "Natural" increasing processes and characterization of predictable times (61-62). P-measures and integrable increasing processes (63-66). Optional and predictable P-measures (67). A Radon-Nikodym theorem (68-68bis). A uniqueness result (69-70). Optional and predictable projections of a P-measure (71-72). Optional and predictable dual projections of a raw increasing process (73-75). Jumps of a

CONTENTS

dual projection (76). Predictable compensator of an optional increasing process (77). Characterization of totally inaccessible stopping times (78). Locally integrable increasing processes (79) and their compensators (80-81). Applications to local martingales (82-85). Random measures (86-88).

3. Increasing processes and potentials.

149

Potential and left potential generated by an increasing process (89). Integration by parts (90-91) and the change of variables formula (92-93). Calculations in L^2 (94-96). Notions on Young functions (97) and the fundamental lemma (98). The Garsia-Neveu inequality (99-100). General form of the maximal inequality (101-104). The case of bounded potentials (105-108). The John-Nirenberg inequality (109). Domination inequalities (110-113).

CHAPTER VII. DECOMPOSITION OF SUPERMARTINGALES, APPLICATIONS

183

1. The decomposition theorem.

183

A functional analytic lemma (1-3). Variations (4-5). Decomposition of positive supermartingales of class (D) (6-9), jumps of the associated increasing process (10). Regular strong supermartingales (11). Decomposition of arbitrary cadlag supermartingales (12-13) and jumps of the increasing process (14). Eliminating difficulties at infinity (15). Uniform integrability of associated increasing processes (16). Supermartingales of class (D) and bounded supermartingales (17). Approximation of the associated increasing process: in the weak (18) or strong (19-20) sense; passing from the discrete to the continuous case (21) and using "approximate Laplacians" (22).

Definition and first properties of semimartingales.

Semimartingales and special semimartingales (23-24). Characterization of semimartingales (25). Constructing semimartingales by pasting together (26-27). Application to changes of time (28-29). Semimartingales over $[0, \infty]$ (30). Products of semimartingales (31-32), convex functions of semimartingales (33-34).

212

Explicit decomposition of the product: one factor of finite variation (35-38). Two local martingale factors: the sharp bracket (39-41) and the square bracket (42-43). Extension of the square bracket to semimartingales (44). Preservation of semimartingales under change of law (Girsanov's theorem) (45-47). Explicit decomposition (Girsanov's problem) (48-50). Using the square bracket to characterize special semimartingales (51) and square integrable martingales (52). Kunita-Watanabe inequality (53-54). An inequality of Stricker (55-56). Change of law transforming a semimartingale into a quasimartingale (57-58). Application to filtration changes (59-61) and to a simple problem of enlarging σ -fields (62-63). Complement to nos. 57-58 (63bis).

3. H^p spaces of martingales and semimartigales The space R^p of cadlag processes, the subspace H^p or M^p of martingales (maximal norm) (64). Dual of R^p , p > 1 (65-66) and p = 1 (67). Representation of increasing processes (68-69). Dual of H^1 , first representation (70). Lemmas about maximal H^1 (71-73). Dual of maximal H^1 (74-75). The spaces BMO_n (76), complete statement of the duality theorem (77), characterization of BMO (78), computing the duality. Some examples of elements of BMO (80). Quadratic H^p spaces (81-82). Lemmas on H^1 (83-85). Fefferman's inequality (86), variations (87). The dual of quadratic H^1 is BMO (88-89). Davis's inequality (90-91), the Burkholder-Davis-Gundy inequality (92). Inequalities for compensators (93-95). The space R⁰ of cadlag processes (96). Localization and prelocalization (97). The space H^p of semimartingales or the space S^p $(p \ge 1)$ (98). Local convergence in S^p (99). The space S^0 of semimartingales up to infinity (100) and the semimartingale topology (101). Complements using the stochastic integral (104-105).

252

309

309

1. Stochastic integral of locally bounded predictable processes Notation for elementary s.i. (1-2). The fundamental theorem for constructing a s.i. (3-6). Comment (7). Extension to locally bounded predictable processes (8-11). Invariance under change of law (12) and filtration (13). A dominated convergence theorem (14). Application to "Riemann sums" (15) and square bracket theory (16-17). Integration by parts formulae (18-19). Invariance of the square bracket under change of law (20-21). The square bracket and s.i. (22). The local character of the s.i. (23-24). The formula for change of variables (25-28) and Tanaka's formula (29).

2. Structure of martingales and local martingales.

Return to square integrable martingales: characterization of the s.i. using the square bracket (30-31). The compensated s.i. (32-35). Extension to local martingales (36-40). Orthogonality of local martingales (41). Compensated sums of jumps and decomposition of a local martingale (42-44). The continuous martingale part of a semimartingale (45). Stable subspaces of M^2 (46-48). Projection theorem in M^2 (49-52). Some examples (53-54). Stable subspaces in M^1 (55-56). Extremal laws and the predictable representation (57-58). The case of Brownian motion (59-63). Processes depending on a parameter and random measures (64), projection theorem for measures (65-66). Application to point processes (67). Lévy decomposition of a local martingale (68).

3. Two extensions of the notion of a stochastic integral. Uncompensated stochastic integrals of optional processes (69-73). Integrals of predictable processes which are not locally bounded (74-77). Vector s.i. and stable subspaces in S (78).

339

377

xii CONTENTS

4. A characterization of semimartingales. The main theorem: statement and preliminaries (79-81). Reduction to a theorem in functional analysis (82). Proof of this (83-85).	386
APPENDIX 1. STRONG SUPERMARTINGALES	393
Definition of optional (1) and predictable (2) strong martingales and supermartingales. Some inequalities (3). Existence of right and left limits (4-5). The projection theorems without the usual conditions (6-8). Two applications (9-10). Dual projections (11-12). Regularity and right continuity in expectation (13). Split stopping times (14) and extension of the stopping theorem (15) and the inequalities referring to increasing processes and potentials (16-19). Mertens decomposition (20-21). Snell envelope (22-23). Characterization of processes of class (D) (24-25).	
APPENDIX 2. COMPLEMENTS ON QUASIMARTINGALES	421
Convex functions and quasimartingales (1). Another expression for the variation (2-3). Quasimartingales of class (D) and decomposition (4). Quasimartingales and finitely additive measures (5).	
COMMENTS	427
BIBLIOGRAPHY	435
INDEX	455
INDEX OF NOTATION	461