

---

# Contents

## 1 Events and their probabilities

- 1.1 Introduction 1
- 1.2 Events as sets 1
- 1.3 Probability 4
- 1.4 Conditional probability 8
- 1.5 Independence 13
- 1.6 Completeness and product spaces 14
- 1.7 Worked examples 16
- 1.8 Problems 21

## 2 Random variables and their distributions

- 2.1 Random variables 26
- 2.2 The law of averages 30
- 2.3 Discrete and continuous variables 33
- 2.4 Worked examples 35
- 2.5 Random vectors 38
- 2.6 Monte Carlo simulation 41
- 2.7 Problems 43

## 3 Discrete random variables

- 3.1 Probability mass functions 46
- 3.2 Independence 48
- 3.3 Expectation 50
- 3.4 Indicators and matching 56
- 3.5 Examples of discrete variables 60
- 3.6 Dependence 62
- 3.7 Conditional distributions and conditional expectation 67
- 3.8 Sums of random variables 70
- 3.9 Simple random walk 71
- 3.10 Random walk: counting sample paths 75
- 3.11 Problems 83

## 4 Continuous random variables

- 4.1 Probability density functions 89
- 4.2 Independence 91
- 4.3 Expectation 93
- 4.4 Examples of continuous variables 95
- 4.5 Dependence 98
- 4.6 Conditional distributions and conditional expectation 104
- 4.7 Functions of random variables 107
- 4.8 Sums of random variables 113
- 4.9 Multivariate normal distribution 115
- 4.10 Distributions arising from the normal distribution 119
- 4.11 Sampling from a distribution 122
- 4.12 Coupling and Poisson approximation 127
- 4.13 Geometrical probability 133
- 4.14 Problems 140

## 5 Generating functions and their applications

- 5.1 Generating functions 148
- 5.2 Some applications 156
- 5.3 Random walk 162
- 5.4 Branching processes 171
- 5.5 Age-dependent branching processes 175
- 5.6 Expectation revisited 178
- 5.7 Characteristic functions 181
- 5.8 Examples of characteristic functions 186
- 5.9 Inversion and continuity theorems 189
- 5.10 Two limit theorems 193
- 5.11 Large deviations 201
- 5.12 Problems 206

## 6 Markov chains

- 6.1 Markov processes 213
- 6.2 Classification of states 220
- 6.3 Classification of chains 223
- 6.4 Stationary distributions and the limit theorem 227
- 6.5 Reversibility 237
- 6.6 Chains with finitely many states 240
- 6.7 Branching processes revisited 243
- 6.8 Birth processes and the Poisson process 246
- 6.9 Continuous-time Markov chains 256
- 6.10 Uniform semigroups 266
- 6.11 Birth–death processes and imbedding 268
- 6.12 Special processes 274
- 6.13 Spatial Poisson processes 281
- 6.14 Markov chain Monte Carlo 291
- 6.15 Problems 296

## 7 Convergence of random variables

- 7.1 Introduction 305
- 7.2 Modes of convergence 308
- 7.3 Some ancillary results 318
- 7.4 Laws of large numbers 325
- 7.5 The strong law 329
- 7.6 The law of the iterated logarithm 332
- 7.7 Martingales 333
- 7.8 Martingale convergence theorem 338
- 7.9 Prediction and conditional expectation 343
- 7.10 Uniform integrability 350
- 7.11 Problems 354

## 8 Random processes

- 8.1 Introduction 360
- 8.2 Stationary processes 361
- 8.3 Renewal processes 365
- 8.4 Queues 367
- 8.5 The Wiener process 370
- 8.6 Existence of processes 371
- 8.7 Problems 373

## 9 Stationary processes

- 9.1 Introduction 375
- 9.2 Linear prediction 377
- 9.3 Autocovariances and spectra 380
- 9.4 Stochastic integration and the spectral representation 387
- 9.5 The ergodic theorem 393
- 9.6 Gaussian processes 405
- 9.7 Problems 409

## 10 Renewals

- 10.1 The renewal equation 412
- 10.2 Limit theorems 417
- 10.3 Excess life 421
- 10.4 Applications 423
- 10.5 Renewal–reward processes 431
- 10.6 Problems 437

## 11 Queues

- 11.1 Single-server queues 440
- 11.2 M/M/1 442
- 11.3 M/G/1 445
- 11.4 G/M/1 451
- 11.5 G/G/1 455

- 11.6 Heavy traffic 462
- 11.7 Networks of queues 462
- 11.8 Problems 468

## 12 Martingales

- 12.1 Introduction 471
- 12.2 Martingale differences and Hoeffding's inequality 476
- 12.3 Crossings and convergence 481
- 12.4 Stopping times 487
- 12.5 Optional stopping 491
- 12.6 The maximal inequality 496
- 12.7 Backward martingales and continuous-time martingales 499
- 12.8 Some examples 503
- 12.9 Problems 508

## 13 Diffusion processes

- 13.1 Introduction 513
- 13.2 Brownian motion 514
- 13.3 Diffusion processes 516
- 13.4 First passage times 525
- 13.5 Barriers 530
- 13.6 Excursions and the Brownian bridge 534
- 13.7 Stochastic calculus 537
- 13.8 The Itô integral 539
- 13.9 Itô's formula 544
- 13.10 Option pricing 547
- 13.11 Passage probabilities and potentials 554
- 13.12 Problems 561

**Appendix I.** Foundations and notation 564

**Appendix II.** Further reading 569

**Appendix III.** History and varieties of probability 571

**Appendix IV.** John Arbuthnot's Preface to *Of the laws of chance* (1692) 573

**Appendix V.** Table of distributions 576

**Appendix VI.** Chronology 578

**Bibliography** 580

**Notation** 583

**Index** 585