

Contents

Preface	vii
Acknowledgments	ix
List of Figures	xxiii
List of Tables	xxvii
1 Introduction	1
1.1 Outline of Book	3
1.1.1 Outline of Part I	3
1.1.2 Outline of Part II	5
1.1.3 Applied Probability and Performance Modeling	6
1.2 Queueing Systems**	8
1.2.1 More General Queueing Systems	9
1.2.2 Queueing Networks	10
I Probability	15
2 Randomness and Probability	17
2.1 Probability: Frequency-Based	20

2.2	Probability: Axiomatic-Based	22
2.2.1	Probability and Modeling	23
2.2.2	Set Operations	26
2.2.3	Probability Spaces	28
2.3	Probability: Immediate Consequences of the Axioms	32
2.3.1	Conditional Probability	34
2.3.2	Independence	36
2.3.3	Causation	38
2.3.4	Bayes' Rule	38
2.4	Summary of Chapter 2	39
2.4.1	Postscript: Calculating π to a Billion Places	41
2.5	Problems for Chapter 2	42

3 Combinatorics 45

3.1	Different Counting Paradigms	47
3.1.1	Counting Methodology	47
3.2	The Four Counting Paradigms	49
3.2.1	Permutations without Replacement	49
3.2.2	Permutations with Replacement	50
3.2.3	Coincidences	52
3.2.4	Combinations without Replacement	54
3.2.5	Problems Equivalent to Combinations without Replacement	56
3.2.6	Basic Properties of Binomial Coefficients	58
3.2.7	Binomial Identities	59
3.2.8	Multinomial Coefficients	61
3.2.9	Partial Sum of Binomial Coefficients	62
3.2.10	Combinations with Replacement	62
3.2.11	Simple Derivation for Combinations with Replacement	64
3.2.12	Problems Equivalent to Combinations with Replacement	65
3.2.13	Basic Properties of Binomial-R Coefficients	66
3.3	Generating Functions	67

3.3.1	Generating Functions for Combinations	68
3.3.2	General Solution for Combinations	69
3.3.3	Algebraic Properties of Generating Functions	74
3.3.4	Addition and Convolution	74
3.3.5	First Passage Times	76
3.4	Solving Recurrence Equations with Generating Functions**	80
3.4.1	An Example Using Generating Functions	82
3.4.2	General Generating Function Solution Method	84
3.4.3	Solutions without Explicitly Determining the Coefficients	88
3.5	Summary of Chapter 3	90
3.6	Problems for Chapter 3	92
4	Random Variables and Distributions	101
4.1	Random Variables	102
4.1.1	Forms of Equality	106
4.1.2	Independence	106
4.1.3	Examples of Random Variables	107
4.1.4	Joint Random Variables	109
4.2	Distribution Functions	111
4.2.1	Impulse Functions	114
4.2.2	Probability Generating Functions	116
4.2.3	Existence of Probability Spaces for a Given Distribution	117
4.3	Unified Derivation of Common Distributions	118
4.3.1	Indicator Random Variables	119
4.3.2	Bernoulli Random Variables	120
4.3.3	Discrete Uniform Distribution	121
4.3.4	Properties of Indicator Random Variables	122
4.4	Constructing Discrete Distributions from Indicator Random Variables	123
4.4.1	Operations Used to Form Random Variables	124
4.4.2	Discrete Uniform Random Variables	124
4.4.3	Binomial Random Variables	125

4.4.4	Sum of Binomial Random Variables with the Same Parameter	127
4.4.5	Intuitive Argument for Sum of Binomial Random Variables	127
4.4.6	Sum of Binomial Random Variables with Different Parameters	128
4.4.7	Geometric Random Variables	129
4.4.8	Geometric Sum of Geometric Random Variables	130
4.4.9	Starting Geometric Distributions from 0	131
4.4.10	Memoryless Property of the Geometric Distribution	131
4.4.11	Discrete + Memoryless \Rightarrow Geometric	132
4.4.12	Negative Binomial Distribution	133
4.4.13	Sum of Negative Binomial Random Variables	135
4.4.14	Starting Negative Binomial Distributions from 0	135
4.4.15	Relationship Between Geometric and Binomial Distributions	136
4.4.16	Renewal Counting Process of Geometric Random Variables	137
4.4.17	Intuitive Argument for the Renewal Counting Process	137
4.4.18	Relationship Between Uniform and Binomial Distributions	139
4.5	Limiting Distributions	140
4.5.1	Continuous Uniform Distribution	140
4.5.2	Probability of Zero \nrightarrow Impossibility	141
4.5.3	The Poisson Distribution	144
4.5.4	Poisson Approximation of a Binomial Distribution	145
4.5.5	Sum of Poisson Random Variables	146
4.5.6	Generalized Limit that Leads to the Poisson Distribution	147
4.5.7	Difference Between Binomial and Poisson Random Variables	148
4.5.8	The Exponential Distribution	149
4.5.9	Continuous + Memoryless \Rightarrow Exponential	151
4.5.10	Geometric Sum of Exponential Random Variables	152
4.5.11	Renewal Counting Process of Exponential Random Variables	158
4.6	Summary of Chapter 4	159
4.7	Problems for Chapter 4	161

5	Expectation and Fundamental Theorems	167
5.1	Expectation	168
5.1.1	Properties of Expectation	169
5.1.2	Expectation of Common Distributions	171
5.1.3	Examples of Random Variables that Do Not Have Expectations	174
5.1.4	Expectation of a Function	175
5.1.5	Jensen's Inequality	176
5.1.6	Expectation of Products of Random Variables	180
5.1.7	Survivor Representation of Expectation	181
5.1.8	Stopping Times	181
5.1.9	Wald's Equality	184
5.2	Expectations of Functions of Random Variables	186
5.2.1	Variance	189
5.2.2	Relationship Between Expectation and Variance	190
5.2.3	Covariance	191
5.3	Moment Generating Functions	193
5.3.1	Probability Generating Functions	195
5.3.2	Moment Generating Functions Determine Densities	197
5.4	The Laplace Transform	197
5.4.1	Common Transforms	198
5.4.2	Properties of the Laplace Transform	199
5.4.3	Solving Differential Equations with Laplace Transforms	203
5.5	Fundamental Theorems	204
5.5.1	Markov's Inequality	205
5.5.2	Chebyshev's Inequality	206
5.5.3	Chernoff's Bound	207
5.5.4	Weak Law of Large Numbers	210
5.5.5	Strong Law of Large Numbers**	212
5.5.6	The Central Limit Theorem	215
5.6	Summary of Chapter 5	219
5.6.1	Ending Historical Notes	220
5.7	Problems for Chapter 5	225

II Stochastic Processes

233

6	The Poisson Process and Renewal Theory	235
6.1	Binomial Processes	237
6.1.1	Basic Properties of a Binomial Process	238
6.1.2	Residual and Age Distributions for Binomial Processes	239
6.1.3	Randomly Selected Times	239
6.1.4	Derivation of the Joint Distribution	240
6.1.5	Length Biasing	242
6.2	Poisson Processes	244
6.2.1	Basic Properties of a Poisson Process	245
6.2.2	Relaxing Stationarity: Nonhomogenous Poisson Processes	246
6.2.3	Relaxing Single Arrivals: Compound Poisson Processes	247
6.2.4	Relaxing Independent Increments: Modulated Poisson Processes	248
6.2.5	Generalized Limit that Leads to the Poisson Process	249
6.2.6	Residual and Age Distributions for Poisson Processes	250
6.2.7	Superposition and Splitting of Poisson Processes	252
6.3	Renewal Processes	254
6.3.1	Limiting Number of Renewals per Unit Time	257
6.3.2	Expected Number of Renewals	257
6.3.3	Renewal Density	260
6.3.4	Limiting Renewal Density	261
6.3.5	Limiting Expected Number of Renewals per Unit Time	262
6.3.6	Nonstationary and Nonindependent Increments	263
6.3.7	Stationary and Independent Increments \Rightarrow Poisson Process	264
6.4	Equilibrium Renewal Processes	265
6.5	Joint Distribution of the Age and Residual Life	267
6.5.1	Stationary Process for all Time	268
6.5.2	Moments for the Age and Residual Life	269
6.5.3	Length Biasing for General Renewal Processes	269
6.6	Alternating Renewal Processes	270

6.6.1	Time Averages	273
6.7	Generalizations for Lattice Random Variables	274
6.8	Summary of Chapter 6	275
6.9	Problems for Chapter 6	278
7	The M/G/1 Queue	283
7.1	Elementary Systems Theory	284
7.1.1	Time Averages and Customer Averages	285
7.1.2	Almost Sure Limits	285
7.1.3	Little's Law	287
7.1.4	Level Crossing Law	291
7.1.5	Poisson Arrivals See Time Averages	293
7.2	Expected Values for the M/G/1 Queue	295
7.2.1	Utilization of the M/G/1 Queue	296
7.2.2	Expected Waiting and Response Times for the M/G/1 Queue	296
7.2.3	General Observations on Queueing Systems	300
7.2.4	Busy Period for M/G/1 Queues	302
7.3	Distribution of Queue Length for the M/G/1 Queue	304
7.3.1	General Observations on the Pollaczek–Khinchin Equation	307
7.3.2	Distribution for Common M/G/1 Queues	309
7.3.3	The M/M/1 Queue	309
7.3.4	The M/ $H_2(k)$ /1 Queue	310
7.3.5	The M/D/1 Queue	312
7.3.6	The M/ E_k /1 Queue	313
7.3.7	Distribution of Response Time for M/G/1 Queues	313
7.4	Distribution of the Number of Customers in a Busy Period	316
7.4.1	Distribution of N_b for the M/M/1 Queue	317
7.5	Summary of Chapter 7	320
7.6	Problems for Chapter 7	324

8	Markov Processes	329
8.1	Markov Chains	333
8.1.1	The Chapman–Kolmogorov Equations	334
8.1.2	Classification of States	337
8.1.3	Mathematical Classification	339
8.1.4	The Stationary Distribution of Ergodic Chains	340
8.1.5	Balance Equations for Stationary Probabilities	343
8.1.6	Uniqueness of Stationary Probabilities	344
8.1.7	Guessing Stationary State Probabilities	348
8.1.8	A Note on Periodic Chains	349
8.2	Global Balance Equations	350
8.3	Semi-Markov Processes	352
8.3.1	Elimination of Self-Loops in Markov Chains	355
8.3.2	Semi-Markov Processes Are Not Markovian	356
8.4	Markov Processes	356
8.4.1	The Generator Matrix	356
8.4.2	The Stationary Distribution of a Markov Process	357
8.5	Variations of the M/M/1 Queue	370
8.5.1	The M/M/1/k Queue	371
8.5.2	The M/M/k Queue	371
8.5.3	The M/M/k/k Queue	372
8.5.4	The M/M/1//p Queue	372
8.6	The G/M/1 Queue	373
8.7	Uniformized Markov Processes	376
8.8	The Chapman–Kolmogorov Differential Equations	378
8.9	Summary of Chapter 8	382
8.10	Problems for Chapter 8	384

9	Matrix Geometric Solutions	391
9.1	Matrix Geometric Systems	392
9.1.1	A Matrix Geometric Example	393
9.1.2	Quasi Birth–Death Processes	397
9.2	General Matrix Geometric Solutions	399
9.3	Matrix Geometric Solutions for Markov Chains	405
9.3.1	Uniformization for Matrix Geometric Markov Processes	406
9.4	Properties of Solutions	409
9.4.1	Stability	409
9.4.2	Interpretation of the Rate Matrix	411
9.4.3	Computational Properties	417
9.4.4	Specifying Matrices	420
9.5	Phase Distributions	421
9.5.1	Continuous Phase Distributions	421
9.5.2	The Distribution of a Continuous Phase Distribution	422
9.5.3	The PH-Markov Process and PH-Renewal Process	423
9.5.4	Discrete Phase Distributions	424
9.6	Summary of Chapter 9	424
9.7	Problems for Chapter 9	426
 10	 Queueing Networks	 429
10.1	Reversibility	431
10.1.1	Reversible Processes	433
10.1.2	The Reversed Process	434
10.1.3	Detailed Balance	438
10.1.4	Stationary Distribution of a Reversible Process	440
10.1.5	Kolmogorov’s Criterion	440
10.2	Properties of Reversible Processes	441
10.2.1	Scaling of Transition Rates	441
10.2.2	Spanning Tree Representation of a Reversible Process	442
10.2.3	Joining of Two Independent Reversible Processes	442

10.2.4	State Truncation in Reversible Processes	443
10.3	Input–Output Properties	445
10.3.1	Input–Output Properties of Reversible Queues	445
10.3.2	Tandem Reversible Queues	446
10.3.3	The Tandem Queue Is Not Reversible	449
10.3.4	Poisson Arrivals and Departures \neq Product Form	450
10.3.5	Features of Product Form Networks	455
10.3.6	Input–Output Properties of Quasi-Reversible Queues	456
10.3.7	Quasi-Reversibility and Reversibility	457
10.3.8	Multiple-Class Quasi-Reversible M/M/1 Queue	458
10.4	Symmetric Queues	460
10.4.1	Quasi-Reversible Symmetric Queues	462
10.4.2	Distribution of Number of Customers in a Symmetric Queue	465
10.5	Partial Balance and Quasi-Reversible Queues	468
10.5.1	Local Balance	469
10.5.2	General Definition of Partial Balance	469
10.5.3	State Truncation Property	470
10.5.4	Summary of Reversibility and Quasi-Reversibility	471
10.6	Properties of Partial Balance and Product Form Networks	472
10.6.1	Description of the Network	472
10.6.2	Markov Routing Policy	473
10.6.3	Types of Transitions	474
10.6.4	The Distribution Property	476
10.6.5	Open Product Form Networks	477
10.6.6	Transition Rates for the Reversed Process	477
10.6.7	Distribution of Number in Queue	481
10.6.8	Distribution of Chains	482
10.6.9	Closed Product Form Networks	483
10.6.10	Calculating the Normalization Constant	486
10.6.11	The Arrival–Departure Property	488
10.6.12	Arrival–Departure Property for Closed Quasi-Reversible Queues	490

10.6.13 Mean Value Analysis	490
10.6.14 The State Aggregation Property	492
10.6.15 Flow Equivalent Servers	494
10.7 Summary of Chapter 10	494
10.8 Problems for Chapter 10	498
11 Epilogue and Special Topics	503
11.1 Stochastic Comparisons	504
11.2 Large Deviation Theory	506
11.3 Bounds on the Response Time for a G/G/1 Queue	510
A Types of Randomness	513
A.1 Randomness: Physical Systems	514
A.1.1 Intrinsic Probability	516
A.2 Randomness: Deterministic Systems	518
A.2.1 The Baker's Transformation	518
A.2.2 Dynamical Systems	521
A.3 Deterministic Randomness**	522
A.3.1 Isomorphism Between Systems	523
A.3.2 Random Newtonian Systems	524
A.4 Summary of Appendix A	525
A.5 Problems for Appendix A	527
B Combinatorial Equalities and Inequalities	529
B.1 Noninteger Combinatorial Expressions	529
B.2 Binomial Formula	530
B.3 Stirling's (de Moivre's) Formula	530
B.4 Bounds on Factorial Expressions	531
B.5 Noninteger Factorials**	532
C Tables of Laplace Transforms and Generating Functions	535
C.0.1 Laplace Transforms	535
C.1 Generating Functions	539

D Limits and Order Relationships	545
D.1 Limits	545
D.2 Order Relationships	547
E List of Common Summations	549
Bibliography	553
References	559
Index of Notation	567
Index	577