

# Contents

<b>Preface</b>	<b>xi</b>
<b>1 Sets and Events</b>	<b>1</b>
1.1 Introduction . . . . .	1
1.2 Basic Set Theory . . . . .	2
1.2.1 Indicator functions . . . . .	5
1.3 Limits of Sets . . . . .	6
1.4 Monotone Sequences . . . . .	8
1.5 Set Operations and Closure . . . . .	11
1.5.1 Examples . . . . .	13
1.6 The $\sigma$ -field Generated by a Given Class $\mathcal{C}$ . . . . .	15
1.7 Borel Sets on the Real Line . . . . .	16
1.8 Comparing Borel Sets . . . . .	18
1.9 Exercises . . . . .	20
<b>2 Probability Spaces</b>	<b>29</b>
2.1 Basic Definitions and Properties . . . . .	29
2.2 More on Closure . . . . .	35
2.2.1 Dynkin's theorem . . . . .	36
2.2.2 Proof of Dynkin's theorem . . . . .	38
2.3 Two Constructions . . . . .	40
2.4 Constructions of Probability Spaces . . . . .	42
2.4.1 General Construction of a Probability Model . . . . .	43
2.4.2 Proof of the Second Extension Theorem . . . . .	49

2.5	Measure Constructions . . . . .	57
2.5.1	Lebesgue Measure on $(0, 1]$ . . . . .	57
2.5.2	Construction of a Probability Measure on $\mathbb{R}$ with Given Distribution Function $F(x)$ . . . . .	61
2.6	Exercises . . . . .	63
<b>3</b>	<b>Random Variables, Elements, and Measurable Maps</b>	<b>71</b>
3.1	Inverse Maps . . . . .	71
3.2	Measurable Maps, Random Elements, Induced Probability Measures . . . . .	74
3.2.1	Composition . . . . .	77
3.2.2	Random Elements of Metric Spaces . . . . .	78
3.2.3	Measurability and Continuity . . . . .	80
3.2.4	Measurability and Limits . . . . .	81
3.3	$\sigma$ -Fields Generated by Maps . . . . .	83
3.4	Exercises . . . . .	85
<b>4</b>	<b>Independence</b>	<b>91</b>
4.1	Basic Definitions . . . . .	91
4.2	Independent Random Variables . . . . .	93
4.3	Two Examples of Independence . . . . .	95
4.3.1	Records, Ranks, Renyi Theorem . . . . .	95
4.3.2	Dyadic Expansions of Uniform Random Numbers . . . . .	98
4.4	More on Independence: Groupings . . . . .	100
4.5	Independence, Zero-One Laws, Borel-Cantelli Lemma . . . . .	102
4.5.1	Borel-Cantelli Lemma . . . . .	102
4.5.2	Borel Zero-One Law . . . . .	103
4.5.3	Kolmogorov Zero-One Law . . . . .	107
4.6	Exercises . . . . .	110
<b>5</b>	<b>Integration and Expectation</b>	<b>117</b>
5.1	Preparation for Integration . . . . .	117
5.1.1	Simple Functions . . . . .	117
5.1.2	Measurability and Simple Functions . . . . .	118
5.2	Expectation and Integration . . . . .	119
5.2.1	Expectation of Simple Functions . . . . .	119
5.2.2	Extension of the Definition . . . . .	122
5.2.3	Basic Properties of Expectation . . . . .	123
5.3	Limits and Integrals . . . . .	131
5.4	Indefinite Integrals . . . . .	134
5.5	The Transformation Theorem and Densities . . . . .	135
5.5.1	Expectation is Always an Integral on $\mathbb{R}$ . . . . .	137
5.5.2	Densities . . . . .	139
5.6	The Riemann vs Lebesgue Integral . . . . .	139
5.7	Product Spaces, Independence, Fubini Theorem . . . . .	143

5.8	Probability Measures on Product Spaces . . . . .	147
5.9	Fubini's theorem . . . . .	149
5.10	Exercises . . . . .	155
<b>6</b>	<b>Convergence Concepts</b>	<b>167</b>
6.1	Almost Sure Convergence . . . . .	167
6.2	Convergence in Probability . . . . .	169
6.2.1	Statistical Terminology . . . . .	170
6.3	Connections Between a.s. and i.p. Convergence . . . . .	171
6.4	Quantile Estimation . . . . .	178
6.5	$L_p$ Convergence . . . . .	180
6.5.1	Uniform Integrability . . . . .	182
6.5.2	Interlude: A Review of Inequalities . . . . .	186
6.6	More on $L_p$ Convergence . . . . .	189
6.7	Exercises . . . . .	195
<b>7</b>	<b>Laws of Large Numbers and Sums of Independent Random Variables</b>	<b>203</b>
7.1	Truncation and Equivalence . . . . .	203
7.2	A General Weak Law of Large Numbers . . . . .	204
7.3	Almost Sure Convergence of Sums of Independent Random Variables . . . . .	209
7.4	Strong Laws of Large Numbers . . . . .	213
7.4.1	Two Examples . . . . .	215
7.5	The Strong Law of Large Numbers for IID Sequences . . . . .	219
7.5.1	Two Applications of the SLLN . . . . .	222
7.6	The Kolmogorov Three Series Theorem . . . . .	226
7.6.1	Necessity of the Kolmogorov Three Series Theorem . . . . .	230
7.7	Exercises . . . . .	234
<b>8</b>	<b>Convergence in Distribution</b>	<b>247</b>
8.1	Basic Definitions . . . . .	247
8.2	Scheffé's lemma . . . . .	252
8.2.1	Scheffé's lemma and Order Statistics . . . . .	255
8.3	The Baby Skorohod Theorem . . . . .	258
8.3.1	The Delta Method . . . . .	261
8.4	Weak Convergence Equivalences; Portmanteau Theorem . . . . .	263
8.5	More Relations Among Modes of Convergence . . . . .	267
8.6	New Convergences from Old . . . . .	268
8.6.1	Example: The Central Limit Theorem for m-Dependent Random Variables . . . . .	270
8.7	The Convergence to Types Theorem . . . . .	274
8.7.1	Application of Convergence to Types: Limit Distributions for Extremes . . . . .	278
8.8	Exercises . . . . .	282

<b>9</b>	<b>Characteristic Functions and the Central Limit Theorem</b>	<b>293</b>
9.1	Review of Moment Generating Functions and the Central Limit Theorem . . . . .	294
9.2	Characteristic Functions: Definition and First Properties . . . . .	295
9.3	Expansions . . . . .	297
9.3.1	Expansion of $e^{ix}$ . . . . .	297
9.4	Moments and Derivatives . . . . .	301
9.5	Two Big Theorems: Uniqueness and Continuity . . . . .	302
9.6	The Selection Theorem, Tightness, and Prohorov's theorem . . . . .	307
9.6.1	The Selection Theorem . . . . .	307
9.6.2	Tightness, Relative Compactness, and Prohorov's theorem . . . . .	309
9.6.3	Proof of the Continuity Theorem . . . . .	311
9.7	The Classical CLT for iid Random Variables . . . . .	312
9.8	The Lindeberg–Feller CLT . . . . .	314
9.9	Exercises . . . . .	321
<b>10</b>	<b>Martingales</b>	<b>333</b>
10.1	Prelude to Conditional Expectation: The Radon–Nikodym Theorem . . . . .	333
10.2	Definition of Conditional Expectation . . . . .	339
10.3	Properties of Conditional Expectation . . . . .	344
10.4	Martingales . . . . .	353
10.5	Examples of Martingales . . . . .	356
10.6	Connections between Martingales and Submartingales . . . . .	360
10.6.1	Doob's Decomposition . . . . .	360
10.7	Stopping Times . . . . .	363
10.8	Positive Super Martingales . . . . .	366
10.8.1	Operations on Supermartingales . . . . .	367
10.8.2	Upcrossings . . . . .	369
10.8.3	Boundedness Properties . . . . .	369
10.8.4	Convergence of Positive Super Martingales . . . . .	371
10.8.5	Closure . . . . .	374
10.8.6	Stopping Supermartingales . . . . .	377
10.9	Examples . . . . .	379
10.9.1	Gambler's Ruin . . . . .	379
10.9.2	Branching Processes . . . . .	380
10.9.3	Some Differentiation Theory . . . . .	382
10.10	Martingale and Submartingale Convergence . . . . .	386
10.10.1	Krickeberg Decomposition . . . . .	386
10.10.2	Doob's (Sub)martingale Convergence Theorem . . . . .	387
10.11	Regularity and Closure . . . . .	388
10.12	Regularity and Stopping . . . . .	390
10.13	Stopping Theorems . . . . .	392

10.14	Wald's Identity and Random Walks . . . . .	398
10.14.1	The Basic Martingales . . . . .	400
10.14.2	Regular Stopping Times . . . . .	402
10.14.3	Examples of Integrable Stopping Times . . . . .	407
10.14.4	The Simple Random Walk . . . . .	409
10.15	Reversed Martingales . . . . .	412
10.16	Fundamental Theorems of Mathematical Finance . . . . .	416
10.16.1	A Simple Market Model . . . . .	416
10.16.2	Admissible Strategies and Arbitrage . . . . .	419
10.16.3	Arbitrage and Martingales . . . . .	420
10.16.4	Complete Markets . . . . .	425
10.16.5	Option Pricing . . . . .	428
10.17	Exercises . . . . .	429
	<b>References</b>	<b>443</b>
	<b>Index</b>	<b>445</b>