

CONTENTS

Károly Jordan (with a list of his scientific works)	V
Author's Preface to the Hungarian edition	XXVII

CHAPTER I

INTRODUCTION

1. Origin	1
2. Chance	1
2.1. Coincidence. Independence	4
2.2. An objective definition of chance	5
3. Probability	7
3.1. Probability judgement	8
3.2. Probability of a single case	11
3.3. Subjectivity of probability judgements	12
3.4. Improbability and surprisingness	13
3.5. Probability of propositions	16
4. Probability judgement based on experience	17
5. Objective probability	19
6. Defining mathematical probability by means of sets	19
6.1. Infinite sets	23
6.2. Countable sets	24
6.3. Sets of the power of the continuum	24
7. Law of large numbers	26
7.1. Mean	27
7.2. Normal distribution	27
7.3. Law of large numbers	29
7.4. In aposteriori probabilities	30
8. Molecular theory	30
8.1. Principle of least squares	32
9. Probability functions	33
10. Probability of testimonies	35
Bibliography	35

CHAPTER II

MATHEMATICAL TOOLS OF THE CALCULUS OF PROBABILITY

11. Combinatorial analysis	38
11.1. Variations	38
11.2. Combinations	39
11.3. Partitio numerorum	39
11.4. Galilei	40
12. Elements of the Calculus of Finite Differences	41
12.1. The second most important operation of the Calculus of Finite Differences	44
12.2. Some fairly important rules of calculating differences	47
13. Some important integrals occurring in the Calculus of Probability	47
13.1. The Gamma function	54
13.2. The Beta function	56
14. Generating functions	59
14.1. Expansion of generating functions	61
15. Difference equations	62
15.1. Solving difference equations directly	64
15.2. Linear homogeneous difference equation of the first order with variable coefficients	66
15.3. Linear difference equations with constant coefficients	66
15.4. Linear homogeneous difference equations of several variables with constant coefficients	67
16. Factor of discontinuity	68
16.1. In case of continuous variables	72
16.2. Laplace's problem	74
17. Moments	77
17.1. Determining moments by generating functions and determining functions by means of moments	79
17.2. Calculating the binomial moments of the data $y = y(x)$ with respect to $x = 0, 1, 2 \dots N - 1$ i.e. $B_s = \sum_{x=s}^N \binom{x}{s} y(x)$	81
17.3. Expressing means by moments	83
18. Thiele's semiinvariants	84
18.1. Particular cases	87
18.2. Determining the semiinvariants starting from the generating function related to deviations	88
18.3. Expressing Thiele's semiinvariants by power moments	91
18.4. Expressing moments by semiinvariants	92
18.5. Other relations between semiinvariants and moments	93
18.6. Moments and semiinvariants in case of a continuous variable	93
19. Approximation according to the principle of least squares	95
19.1. The principle of least squares	99
19.2. Approximation by orthogonal polynomials	101
20. The principle of moments	102

21. Approximation of degree n by orthogonal moments	106
21.1. Computing the table of approximate values	107
21.2. Computational technique in case of approximating by a straight line	108
21.3. In case of a parabola of the third degree	112
22. Deduction of orthogonal polynomials	113
23. Approximation of observations by exponential function	119
24. Approximation by inverse powers of x	122
25. Approximation of observations following a periodic course by trigonometric functions	124
26. Approximation by Poisson's function $\psi(m, x)$	133
27. Approximation by means of the generalized formula of Poisson	138
27.1 Measuring the attained approximation	144
28. Approximation by a function of continuous variable	144
29. Legendre polynomials	147
30. Hermite polynomials	148
31. Approximation of a function of continuous variable by Poisson's formula	150
31.1. Generalization of the function (1)	154
32. Approximation by the Beta function	157
33. Graduation by orthogonal moments according to the principle of least squares	162
34. Construction of tables	165
35. Linear interpolation	167
35.1. Linear interpolation with a desk calculator	170
36. Interpolation of higher degree	171
36.1. The interpolation formula of Everett	172
37. An interpolation formula in which differences are not needed	173
37.1. Carrying out interpolation of the third degree with a desk calculator	175
37.2. Interpolation of the fifth degree	176
37.3. An example of the interpolation of the third degree	176
38. Interpolation of functions of two and three variables: $f(u, v)$	177
38.1. A similar procedure in case of three independent variables	178
38.2. Interpolation of the third degree in case of two independent variables	178
39. Inverse interpolation	181
40. Determining the accuracy of the interpolation from the tabular data	183
41. Some important tables of probability functions	185
41.1. The table of Laplace's integral	185
41.2. The probability function of Poisson	186
Bibliography	186

CHAPTER III

PROBABILISTIC THEOREMS

42. Elements of set theory	188
42.1. Mathematical definition of probabilities	190
42.2. The theorem of compound probabilities for two cases	190

42.3. The theorem of compound probability for n cases	191
42.4. The theorem of total probability for two cases	194
43. The theorem of general probability	195
43.1. Determining the binomial moments of the general probability function	199
43.2. The second theorem of general probability	199
44. The third theorem of general probability	200
44.1. Duration of the game in case of discontinuous and continuous variables	203
45. Relative probabilities	204
46. The theorem of probabilities of causes or Bayes' theorem	204
46.1. Three cases of Bayes' theorem	207
46.2. Second case of Bayes' theorem	208
46.3. Third case of Bayes' theorem	209
46.4. The second inverse problem	210
46.5. The probability of causes in case of several variables	213
46.6. Application	214
47. The probability of the testimonies	215
48. The probability of a judgement	217
49. The rule of succession	219
Bibliography	221

CHAPTER IV

ARITHMETIC AND GEOMETRIC EXPECTATION

50. Arithmetic expectation	222
50.1. Second definition of the probabilistic judgement	223
50.2. Mathematical expectation and arithmetic mean	224
51. Chebyshev's theorem on the arithmetic expectation	227
51.1. Chebyshev's 3rd theorem	228
51.2. Application of Chebyshev's theorem to squares of deviations from the mean	229
51.3. Application of Chebyshev's theorem to the deviation of sums from their arith-	
metric expectations	230
51.4. A special case	231
51.5. Dicing	231
51.6. The game "heads and tails"	232
51.7. The theorem of errors	232
51.8. Bernoulli's probability function and its generalization due to Poisson obeying	
the law of large numbers	233
52. The impossibility of gambling systems	233
53. The risk	235
54. Geometric expectation	236
54.1. Daniel Bernoulli (1738)	237
54.2. The psychophysical law of Weber and Fechner	238

54.3. Theorems concerning the geometric mean	240
54.4. Examples verified by experience	241
54.5. Investigation of operations from the point of view of geometric expectation	244
Bibliography	247

CHAPTER V

PROBABILITIES CONCERNING REPEATED TRIALS.
CASE OF A SINGLE VARIABLE

55. The first problem of Bernoulli	249
55.1. Moments referred to the number ν of favourable cases belonging to Bernoulli's probability function (1)	252
55.2. Moments referred to the deviations $\xi = \nu - np$ of Bernoulli's function: $M'_s = \sum \xi_0^s P(\nu)$	254
55.3. The semiinvariants of Bernoulli's functions	255
55.4. The mean of the powers of the relative deviations	256
55.5. Determination of the numerical values of Bernoulli's probability function	257
56. The problem of the mean	257
56.1. Determining the probability (4) by means of Dirichlet's discontinuity factor	259
56.2. The inverse problem of the means	260
56.3. The second inverse problem	261
57. The second problem of Bernoulli	262
57.1. Incomplete moments of Bernoulli's function	264
57.2. Central value of median	267
58. The generalized theorem of Simmons Examples	269
59. Approximation of statistical observations on the basis of the principle of moments by means of Bernoulli's formula	272
59.1. Montessus de Ballore's example	274
60. Approximation of Bernoulli's problem by Laplace's function	275
60.1. The principle of moments	277
60.2. Showing the approximation in a graphical way	278
61. Approximation of Bernoulli's second problem by Laplace's function	279
61.1. Approximation of a function of discontinuous variable by some function of continuous variable	281
61.2. Numerical computation	281
61.3. Computation of the values of $L(x)$	282
61.4. Graphical representation	285
62. Application of Laplace's formula	285
62.1. Probability of differences	287
62.2. Probability of the observed dispersion	288
62.3. The inverse problem	290
62.4. Knowing neither the mean nor the dispersion of the multitude	291

62.5. Proof of the fact that Laplace distribution obeys the law of large numbers	292
62.6. The risk in using Laplace's formula	293
63. Application in statistics	294
63.1. Normal distribution	295
63.2. Quetelet's example to the approximation by Laplace's function	296
63.3. Statistical application	297
64. Approximation of Bernoulli's problem by Hermite functions	300
64.1. Laplace's asymmetrical formula	301
64.2. Numerical computation	302
65. Approximation of Bernoulli's probability function by Poisson's function $\psi(m, v)$	304
65.1. Approximation of Bernoulli's second problem	306
65.2. In the case when n observations occur during unit time	307
65.3. On approximation of Bernoulli's function due to Poisson	308
65.4. Application	308
66. Approximation of Bernoulli's probability function by the generalized formula of Poisson	311
67. Inversion of Bernoulli's function	314
67.1. Second inverse problem	315
67.2. Approximation of Bernoulli's inverse probability function by Poisson's formula	316
67.3. From formula (5) we can be obtain the probability of p	318
68. Laplace's approximation of Bernoulli's inverse formula	318
68.1. Second problem	319
68.2. Application in statistics. Comparing a posteriori probabilities	320
69. Poisson's problem of general probability	322
69.1. Semiinvariants of Poisson's probability function	323
69.2. Numerical computation of probabilities	324
69.3. The approximating formula of Poisson	325
69.4. Laplace's approximating formula	325
69.5. The factorial moments of Poisson's function	326
70. Lexis' problem	330
70.1. A special case	332
70.2. Application in statistics	334
71. Hypergeometric probabilities	335
71.1. Inverse problem	337
72. The general case of hypergeometric probabilities	340
72.1. Special case: $\nu = 0$	344
72.2. Approximation of statistical observations by means of hypergeometric probabilities	345
Bibliography	347

CHAPTER VI

PROBABILITIES CONCERNING REPEATED TRIALS
CASE OF SEVERAL VARIABLES

73. Bernoulli's multivariate formula	349
74. Approximation of Bernoulli's problem on $s - 1$ independent variables by means of Poisson's formula	351
75. The multivariate probabilistic formula of Bravais	352
75.1. Moments of first order	354
75.2. Determination of the constant C	355
75.3. Equally probable spherical shells	356
75.4. Application	357
76. A particular case of Bravais' formula	359
76.1. Approximation of Bernoulli's problem of two independent variables by Bravais' formula	361
76.2. Approximation of Bernoulli's formula of $s - 1$ independent variables of Bravais' formula	364
77. Inversion of Bernoulli's formula of $s - 1$ independent variables	365
77.1. Approximation of the inverse function (2) according to the principle of moments	366
77.2. Approximation of Bernoulli's inverse function (2) of $s - 1$ independent variables of Bravais' formula	367
78. Application in statistics	368
78.1. Apriori procedure	369
79. Hypergeometric functions of several variables	371
79.1. Computation of the factorial moments of the probability (1)	371
79.2. Inversion of the hypergeometric probabilities in case of $s - 1$ independent variables	372
80. The general case of hypergeometric probabilities of several variables	374
80.1. Generating function of hypergeometric probabilities of several variables	376
80.2. Approximation by means of Bravais' formula	377
80.3. Inversion of hypergeometric probabilities	378
81. Sampling theory	380
81.1. A set containing favourable and unfavourable elements in an unknown ratio	383
82. Correlations between properties	385
82.1. The apriori problem	386
83. Probability of the real correlation	389
83.1. Determining the probability of the real correlation by means of Poisson's approximation formula	391
83.2. Example	392
83.3. Approximation of the probability of the real correlation, due to Laplace	393
84. General correlation in a table	394
Bibliography	395

CHAPTER VII

PROBABILISTIC PROBLEMS

85. The problem of the ruin or the duration of game	396
85.1. Application	398
86. The third problem of game	399
86.1. Solution of the difference equation by Ellis' method	401
86.2. Transforming formula (5)	403
86.3. Formula (8') deduced for the case of games of odd number	406
86.4. Particular cases	407
86.5. Application	407
87. The second problem of the ruin	411
87.1. Application	413
87.2. Case of a fair game	415
87.3. The mean duration of game until equalization	417
87.4. The ballot problem	418
88. The problem of the moving point	419
89. The problem of match	429
89.1. The duration of play	431
89.2. The more general case of the problem of rencontre	432
89.3. Deduction and solution of the difference equation (17) of Montmort	437
89.4. Statistical application of the problem of rencontre	439
90. The problem of sharing	440
90.1. The case of three players	442
90.2. The more general case of the problem of sharing	444
90.3. Pascal's solution	445
90.4. Problem of three variables	447
91. The Montmort—Moivre problem	449
91.1. Determination of the probability (1) by means of Dirichlet's discontinuity factors	452
91.2. Evaluating the probability using formula (8)	454
91.3. Determining the probability of two urns	455
91.4. Problems on the sum and difference of the drawn numbers	457
91.5. Determining the probability of the ν th drawn number using formula (18)	458
91.6. Determining the probability that the sum will equal z	458
91.7. Determining the probability that the absolute value of the difference of the two drawn numbers will equal z	459
91.8. Determining the probability that the sum x of the drawn numbers will be divisible by a	460
92. Laplace's modification	461
92.1. The number N of terms of some functions of n independent variables	462
92.2. Generalization of Laplace's formula by Kürschák	463
92.3. The problem on drawing m numbers simultaneously	464
92.4. The Montmort—Moivre problem in case of a continuous variable (Laplace)	467
93. Lottery	470

94. Poker	472
94.1. Investigation of the second phase of the game	478
94.2. A particular case	480
94.3. Another case	480
94.4. The case when the player gets a three and two other cards	481
94.5. The case of two pairs	482
94.6. Having none of the enumerated combinations	482
94.7. Playing with the Joker	483
95. Roulette	487
95.1. The system of play	488
95.2. Single occurrence in the simplest algebraic system	488
95.3. A system in which the twofold successive occurrence of an event is needed to win	490
95.4. A playing system that is based on the idea of equalization	492
95.5. The piling up system	493
95.6. The geometric systems	494
96. Mathematical problems of roulette	496
96.1. The probability of obtaining λ red sequences without regard to the observed number of reds in n observations	498
96.2. A particular case in which no red sequence longer than k takes place	498
97. Particular case	500
97.1. The probability of alternations	501
97.2. Determining the probability that there will be isolated observation in n ob- servations	502
97.3. Determining the probability that ν isolated observations will occur in n ob- servations	503
97.4. The arithmetic expectation of the isolated observations	504
97.5. Determining the possibility that in n observations no sequence of length s will occur	505
97.6. Determining the possibility that a sequence of length s will occur ν times in n observations	505
97.7. Arithmetic expectation of the length of the sequences	506
97.8. Determining the most probable distribution	509
97.9. The most probable distribution of the sequences in case of a continuous variable	510
98. Trente et quarante	511
Bibliography	516

CHAPTER VIII

GEOMETRICAL PROBABILITIES

99. Introduction	518
99.1. Probabilities in case of sets of continuum power	520
100. The problem of the points in the one-, two-, and three-dimensional spaces	521
100.1. Determining the probability of the length of the distance between two points	521

100.2. Determining the probability that a triangle can be constructed from a rod of length A	522
100.3. Montmort's problem in case of continuous variables	522
100.4. The two-dimensional problem	523
100.5. The three-dimensional problem	524
101. Straight lines of equal density in the plane	524
101.1. The equation of the convex curve in tangential polar coordinates	525
101.2. Bertrand's problem	526
101.3. Determining the probability that the line cuts the curve	526
101.4. Case of two convex curves outside each other	527
101.5. The density of the points of intersection	528
102. The chords passing through a point of the convex domain	529
102.1. The probability that the chord is longer than g	530
102.2. Particular case	531
102.3. A more general form of Bertrand's problem	533
103. The line segment drawn from a point of the convex domain	535
104. Distribution of straight lines passing through two points of a convex domain	539
104.1. Determining the probability of a line will be longer than g in a particular case	542
104.2. Connecting two points of the convex domain	542
105. Problems concerning boundary points	543
105.1. A chord passing through an inner point and a boundary point of the domain	546
105.2. Connecting a boundary point of the convex domain with an inner point	547
106. The points are not uniformly distributed on a given straight line	547
106.1. Determining the probability of the coordinates of a point	547
Bibliography	549

CHAPTER IX

THE THEORY OF ERRORS

107. The law of errors	550
107.1. The theory of least squares	551
107.2. Knowing the magnitude and dispersion of the results of measurement	552
107.3. Means	553
107.4. The apriori problem	553
107.5. The aposteriori problem	553
107.6. Knowing the true magnitude	554
107.7. Not knowing the true magnitude and dispersion	555
107.8. The probability of the dispersion σ^2 without regard to the true magnitude	557
107.9. Weight and precision	558
108. Indirect measurements	558
108.1. Special case	559
108.2. Determining the most probable values	559

109. Gauss' procedure for solving normal equations and determining the sum of squares of errors	564
109.1. Carrying out the computations and checking	565
109.2. Final checking	566
109.3. The method of unknown multipliers	566
109.4. Example	568
109.5. Computing the unknowns by means of the first method of Gauss	571
109.6. Computing the unknowns by means of the second method of Gauss	572
110. Indirect problem in case of nonlinear equations	573
111. Deduction of the law of errors	574
112. The problem of the average and the sum	576
112.1. Probabilities and dispersions of sums	576
112.2. Probability of the dispersion of errors of measurement	577
113. Transformation of a quadratic form into a sum of squares	578
113.1. Probability of dispersion	581
114. The two-dimensional law of errors	584
114.1. General case of the law of errors of two variables	585
114.2. Bravais' formula	587
Bibliography	588

CHAPTER X

THE THEORY OF GAMES

115. Introduction to the kinetic theory of gases	589
115.1. Probability and mean of velocities	591
115.2. Calculating the probability of the component u	592
116. Determination of the root mean square velocity of the molecules of the gas	592
116.1. Root mean square velocity of the molecules of hydrogen	593
116.2. Laws of gases	593
116.3. Mixtures of gases	595
116.4. Free path	596
116.5. Example	596
117. Probability of the state of gas	597
117.1. Stationary state	598
117.2. The probability of the state of gas from the point of view of velocities	598
117.3. Boltzmann's "H-theorem"	600
118. Entropy	601
119. Another way of treating the state of gases	602
Bibliography	604
Tables of binomial coefficients	605
Appendix	612
Author index	613
Subject index	616