
TABLE OF CONTENTS

PREFACE

TABLE OF CONTENTS

I. INTRODUCTION	1
1. General outline.	1
2. About our notation.	5
2.1. Definitions.	5
2.2. A few important univariate densities.	7
3. Assessment of random variate generators.	8
3.1. Distributions with no variable parameters.	9
3.2. Parametric families.	9
4. Operations on random variables.	11
4.1. Transformations.	11
4.2. Mixtures.	16
4.3. Order statistics.	17
4.4. Convolutions. Sums of independent random variables.	19
4.5. Sums of independent uniform random variables.	21
4.6. Exercises.	23
II. GENERAL PRINCIPLES IN RANDOM VARIATE GENERATION	27
1. Introduction.	27
2. The inversion method.	27
2.1. The inversion principle.	27
2.2. Inversion by numerical solution of $F(X)=U$.	31
2.3. Explicit approximations.	35
2.4. Exercises.	36
3. The rejection method.	40
3.1. Definition.	40
3.2. Development of good rejection algorithms.	43
3.3. Generalizations of the rejection method.	47
3.4. Wald's equation.	50
3.5. Letac's lower bound.	52

3.6. The squeeze principle.	53
3.7. Recycling random variates.	58
3.8. Exercises.	60
4. Decomposition as discrete mixtures.	66
4.1. Definition.	66
4.2. Decomposition into simple components.	66
4.3. Partitions into intervals.	67
4.4. The waiting time method for asymmetric mixtures.	71
4.5. Polynomial densities on $[0,1]$.	71
4.6. Mixtures with negative coefficients.	74
5. The acceptance-complement method.	75
5.1. Definition.	75
5.2. Simple acceptance-complement methods.	77
5.3. Acceleration by avoiding the ratio computation.	78
5.4. An example : nearly flat densities on $[0,1]$.	79
5.5. Exercises.	81
III. DISCRETE RANDOM VARIATES	83
1. Introduction.	83
2. The inversion method.	85
2.1. Introduction.	85
2.2. Inversion by truncation of a continuous random variate.	87
2.3. Comparison-based inversions.	88
2.4. The method of guide tables.	96
2.5. Inversion by correction.	98
2.6. Exercises.	101
3. Table look-up methods.	102
3.1. The table look-up principle.	102
3.2. Multiple table look-ups.	104
4. The alias method.	107
4.1. Definition.	107
4.2. The alias-urn method.	110
4.3. Geometrical puzzles.	111
4.4. Exercises.	112
5. Other general principles.	113
5.1. The rejection method.	113
5.2. The composition and acceptance-complement methods.	116
5.3. Exercises.	116
IV. SPECIALIZED ALGORITHMS	118
1. Introduction.	118
1.1. Motivation for the chapter.	118
1.2. Exercises.	118
2. The Forsythe-von Neumann method.	121
2.1. Description of the method.	121
2.2. Von Neumann's exponential random variate generator.	125
2.3. Monahan's generalization.	127

2.4. An example : Vaduva's gamma generator.	130
2.5. Exercises.	132
3. Almost-exact inversion.	133
3.1. Definition.	133
3.2. Monotone densities on $[0, \infty)$.	134
3.3. Polya's approximation for the normal distribution.	135
3.4. Approximations by simple functions of normal random varlates.	136
3.5. Exercises.	143
4. Many-to-one transformations.	145
4.1. The principle.	145
4.2. The absolute value transformation.	147
4.3. The Inverse gaussian distribution.	148
4.4. Exercises.	150
5. The series method.	151
5.1. Description.	151
5.2. Analysis of the alternating series algorithm.	154
5.3. Analysis of the convergent series algorithm.	156
5.4. The exponential distribution.	157
5.5. The Raab-Green distribution.	158
5.6. The Kolmogorov-Smirnov distribution.	161
5.7. Exercises.	168
6. Representations of densities as integrals.	171
6.1. Introduction.	171
6.2. Khinchine's and related theorems.	171
6.3. The Inverse-of- f method for monotone densities.	178
6.4. Convex densities.	179
6.5. Recursive methods based upon representations.	180
6.6. A representation for the stable distribution.	183
6.7. Densities with Polya type characteristic functions.	186
6.8. Exercises.	191
7. The ratio-of-uniforms method.	194
7.1. Introduction.	194
7.2. Several examples.	197
7.3. Exercises.	203
V. UNIFORM AND EXPONENTIAL SPACINGS	206
1. Motivation.	206
2. Uniform and exponential spacings.	207
2.1. Uniform spacings.	207
2.2. Exponential spacings.	211
2.3. Exercises.	213
3. Generating ordered samples.	213
3.1. Generating uniform $[0,1]$ order statistics.	214
3.2. Bucket sorting. Bucket searching.	215
3.3. Generating exponential order statistics.	219
3.4. Generating order statistics with distribution function F .	220
3.5. Generating exponential random varlates in batches.	223

3.6. Exercises.	223
4. The polar method.	225
4.1. Radially symmetric distributions.	225
4.2. Generating random vectors uniformly distributed on C_d .	230
4.3. Generating points uniformly in and on C_2 .	233
4.4. Generating normal random variates in batches.	235
4.5. Generating radially symmetric random vectors.	236
4.6. The deconvolution method.	239
4.7. Exercises.	240
VI. THE POISSON PROCESS	246
1. The Poisson process.	246
1.1. Introduction.	246
1.2. Simulation of homogeneous Poisson processes.	248
1.3. Nonhomogeneous Poisson processes.	250
1.4. Global methods for nonhomogeneous Poisson process simulation.	257
1.5. Exercises.	258
2. Generation of random variates with a given hazard rate.	260
2.1. Hazard rate. Connection with Poisson processes.	260
2.2. The inversion method.	261
2.3. The composition method.	262
2.4. The thinning method.	264
2.5. DHR distributions. Dynamic thinning.	267
2.6. Analysis of the dynamic thinning algorithm.	269
2.7. Exercises.	276
3. Generating random variates with a given discrete hazard rate.	278
3.1. Introduction.	278
3.2. The sequential test method.	279
3.3. Hazard rates bounded away from 1.	280
3.4. Discrete dynamic thinning.	283
3.5. Exercises.	284
VII. UNIVERSAL METHODS	286
1. Black box philosophy.	286
2. Log-concave densities.	287
2.1. Definition.	287
2.2. Inequalities for log-concave densities.	288
2.3. A black box algorithm.	290
2.4. The optimal rejection algorithm.	293
2.5. The mirror principle.	295
2.6. Non-universal rejection methods.	298
2.7. Exercises.	308
3. Inequalities for families of densities.	310
3.1. Motivation.	310
3.2. Bounds for unimodal densities.	310

3.3. Densities satisfying a Lipschitz condition.	320
3.4. Normal scale mixtures.	325
3.5. Exercises.	328
4. The inversion-rejection method.	331
4.1. The principle.	331
4.2. Bounded densities.	332
4.3. Unimodal and monotone densities.	334
4.4. Monotone densities on $[0,1]$.	335
4.5. Bounded monotone densities : inversion-rejection based on Newton-Raphson iterations.	341
4.6. Bounded monotone densities : geometrically increasing interval sizes.	344
4.7. Lipschitz densities on $[0,\infty)$.	348
4.8. Exercises.	355

VIII. TABLE METHODS FOR CONTINUOUS RANDOM VARIATES

358

1. Composition versus rejection.	358
2. Strip methods.	359
2.1. Definition.	359
2.2. Example 1 : monotone densities on $[0,1]$.	362
2.3. Other examples.	366
2.4. Exercises.	367
3. Grid methods.	368
3.1. Introduction.	368
3.2. Generating a point uniformly in a compact set A .	368
3.3. Avoidance problems.	372
3.4. Fast random variate generators.	375

IX. CONTINUOUS UNIVARIATE DENSITIES

379

1. The normal density.	379
1.1. Definition.	379
1.2. The tail of the normal density.	380
1.3. Composition/rejection methods.	382
1.4. Exercises.	391
2. The exponential density.	392
2.1. Overview.	392
2.2. Marsaglia's exponential generator.	394
2.3. The rectangle-wedge-tail method.	397
2.4. Exercises.	401
3. The gamma density.	401
3.1. The gamma family.	401
3.2. Gamma variate generators.	404
3.3. Uniformly fast rejection algorithms for $a \geq 1$.	407
3.4. The Weibull density.	414
3.5. Johnk's theorem and its implications.	416
3.6. Gamma variate generators when $a \leq 1$.	419
3.7. The tail of the gamma density.	420

3.8. Stacy's generalized gamma distribution.	423
3.9. Exercises.	423
4. The beta density.	428
4.1. Properties of the beta density.	428
4.2. Overview of beta generators.	431
4.3. The symmetric beta density.	433
4.4. Uniformly fast rejection algorithms.	437
4.5. Generators when $\min(a, b) \leq 1$.	439
4.6. Exercises.	444
5. The t distribution.	445
5.1. Overview.	445
5.2. Ordinary rejection methods.	447
5.3. The Cauchy density.	450
5.4. Exercises.	451
6. The stable distribution.	454
6.1. Definition and properties.	454
6.2. Overview of generators.	458
6.3. The Bergstrom-Feller series.	460
6.4. The series method for stable random variates.	463
6.5. Exercises.	467
7. Nonstandard distributions.	468
7.1. Bessel function distributions.	468
7.2. The logistic and hyperbolic secant distributions.	471
7.3. The von Mises distribution.	473
7.4. The Burr distribution.	476
7.5. The generalized inverse gaussian distribution.	478
7.6. Exercises.	480
X. DISCRETE UNIVARIATE DISTRIBUTIONS	485
1. Introduction.	485
1.1. Goals of this chapter.	485
1.2. Generating functions.	486
1.3. Factorials.	489
1.4. A universal rejection method.	493
1.5. Exercises.	496
2. The geometric distribution.	498
2.1. Definition and genesis.	498
2.2. Generators.	499
2.3. Exercises.	500
3. The Poisson distribution.	501
3.1. Basic properties.	501
3.2. Overview of generators.	502
3.3. Simple generators.	502
3.4. Rejection methods.	506
3.5. Exercises.	518
4. The binomial distribution.	520
4.1. Properties.	520

4.2. Overview of generators.	523
4.3. Simple generators.	523
4.4. The rejection method.	526
4.5. Recursive methods.	536
4.6. Symmetric binomial random variates.	538
4.7. The negative binomial distribution.	543
4.8. Exercises.	543
5. The logarithmic series distribution.	545
5.1. Introduction.	545
5.2. Generators.	546
5.3. Exercises.	549
6. The Zipf distribution.	550
6.1. A simple generator.	550
6.2. The Planck distribution.	552
6.3. The Yule distribution.	553
6.4. Exercises.	553
XI. MULTIVARIATE DISTRIBUTIONS	554
1. General principles.	554
1.1. Introduction.	554
1.2. The conditional distribution method.	555
1.3. The rejection method.	557
1.4. The composition method.	557
1.5. Discrete distributions.	559
1.6. Exercises.	562
2. Linear transformations. The multivariate normal distribution.	563
2.1. Linear transformations.	563
2.2. Generators of random vectors with a given covariance matrix.	564
2.3. The multivariate normal distribution.	566
2.4. Points uniformly distributed in a hyperellipsoid.	567
2.5. Uniform polygonal random vectors.	568
2.6. Time series.	571
2.7. Singular distributions.	571
2.8. Exercises.	572
3. Dependence. Bivariate distributions.	573
3.1. Creating and measuring dependence.	573
3.2. Bivariate uniform distributions.	576
3.3. Bivariate exponential distributions.	583
3.4. A case study: bivariate gamma distributions.	586
3.5. Exercises.	588
4. The Dirichlet distribution.	593
4.1. Definitions and properties.	593
4.2. Liouville distributions.	596
4.3. Exercises.	599
5. Some useful multivariate families.	600
5.1. The Cook-Johnson family.	600

5.2. Multivariate Khinchine mixtures.	603
5.3. Exercises.	604
6. Random matrices.	605
6.1. Random correlation matrices.	605
6.2. Random orthogonal matrices.	607
6.3. Random $R \times C$ tables.	608
6.4. Exercises.	610
XII. RANDOM SAMPLING	611
1. Introduction.	611
2. Classical sampling.	612
2.1. The swapping method.	612
2.2. Classical sampling with membership checking	613
2.3. Exercises.	619
3. Sequential sampling.	619
3.1. Standard sequential sampling.	619
3.2. The spacings method for sequential sampling.	621
3.3. The inversion method for sequential sampling.	624
3.4. Inversion-with-correction.	625
3.5. The ghost point method.	626
3.6. The rejection method.	631
3.7. Exercises.	635
4. Oversampling.	635
4.1. Definition.	635
4.2. Exercises.	638
5. Reservoir sampling.	638
5.1. Definition.	638
5.2. The reservoir method with geometric jumps.	640
5.3. Exercises.	641
XIII. RANDOM COMBINATORIAL OBJECTS	642
1. General principles.	642
1.1. Introduction.	642
1.2. The decoding method.	643
1.3. Generation based upon recurrences.	645
2. Random permutations.	648
2.1. Simple generators.	648
2.2. Random binary search trees.	648
2.3. Exercises.	650
3. Random binary trees.	652
3.1. Representations of binary trees.	652
3.2. Generation by rejection.	655
3.3. Generation by sequential sampling.	656
3.4. The decoding method.	657
3.5. Exercises.	657
4. Random partitions.	657
4.1. Recurrences and codewords.	657

4.2. Generation of random partitions.	660
4.3. Exercises.	661
5. Random free trees.	662
5.1. Prufer's construction.	662
5.2. Klingsberg's algorithm.	664
5.3. Free trees with a given number of leaves.	665
5.4. Exercises.	666
6. Random graphs.	667
6.1. Random graphs with simple properties.	667
6.2. Connected graphs.	668
6.3. Tinhofer's graph generators.	669
6.4. Bipartite graphs.	671
6.5. Exercises.	673
XIV. PROBABILISTIC SHORTCUTS AND ADDITIONAL TOPICS	674
1. The maximum of iid random variables.	674
1.1. Overview of methods.	674
1.2. The quick elimination principle.	675
1.3. The record time method.	679
1.4. Exercises.	681
2. Random variates with given moments.	682
2.1. The moment problem.	682
2.2. Discrete distributions.	686
2.3. Unimodal densities and scale mixtures.	687
2.4. Convex combinations.	689
2.5. Exercises.	693
3. Characteristic functions.	695
3.1. Problem statement.	695
3.2. The rejection method for characteristic functions.	696
3.3. A black box method.	700
3.4. Exercises.	715
4. The simulation of sums.	716
4.1. Problem statement.	716
4.2. A detour via characteristic functions.	718
4.3. Rejection based upon a local central limit theorem.	719
4.4. A local limit theorem.	720
4.5. The mixture method for simulating sums.	731
4.6. Sums of independent uniform random variables.	732
4.7. Exercises.	734
5. Discrete event simulation.	735
5.1. Future event set algorithms.	735
5.2. Reeves's model.	738
5.3. Linear lists.	740
5.4. Tree structures.	747
5.5. Exercises.	748
6. Regenerative phenomena.	749
6.1. The principle.	749

6.2. Random walks.	749
6.3. Birth and death processes.	755
6.4. Phase type distributions.	757
6.5. Exercises.	758
7. The generalization of a sample.	759
7.1. Problem statement.	759
7.2. Sample independence.	760
7.3. Consistency of density estimates.	762
7.4. Sample indistinguishability.	763
7.5. Moment matching.	764
7.6. Generators for f_n .	765
7.7. Exercises.	766
XV. THE RANDOM BIT MODEL	768
1. The random bit model.	768
1.1. Introduction.	768
1.2. Some examples.	769
2. The Knuth-Yao lower bound.	771
2.1. DDG trees.	771
2.2. The lower bound.	771
2.3. Exercises.	775
3. Optimal and suboptimal DDG-tree algorithms.	775
3.1. Suboptimal DDG-tree algorithms.	775
3.2. Optimal DDG-tree algorithms.	777
3.3. Distribution-free inequalities for the performance of optimal DDG-tree algorithms.	780
3.4. Exercises.	782
REFERENCES	784
INDEX	817