

Contents

1. A Historical Introduction	
1.1 Motivation	1
1.2 Some Historical Examples	2
1.2.1 Brownian Motion	2
1.2.2 Langevin's Equation	6
1.3 Birth-Death Processes	8
1.4 Noise in Electronic Systems	11
1.4.1 Shot Noise	11
1.4.2 Autocorrelation Functions and Spectra	15
1.4.3 Fourier Analysis of Fluctuating Functions: Stationary Systems	17
1.4.4 Johnson Noise and Nyquist's Theorem	18
2. Probability Concepts	
2.1 Events, and Sets of Events	21
2.2 Probabilities	22
2.2.1 Probability Axioms	22
2.2.2 The Meaning of $P(A)$	23
2.2.3 The Meaning of the Axioms	23
2.2.4 Random Variables	24
2.3 Joint and Conditional Probabilities: Independence	25
2.3.1 Joint Probabilities	25
2.3.2 Conditional Probabilities	25
2.3.3 Relationship Between Joint Probabilities of Different Orders	26
2.3.4 Independence	27
2.4 Mean Values and Probability Density	28
2.4.1 Determination of Probability Density by Means of Arbitrary Functions	28
2.4.2 Sets of Probability Zero	29
2.5 Mean Values	29
2.5.1 Moments, Correlations, and Covariances	30
2.5.2 The Law of Large Numbers	30
2.6 Characteristic Function	32
2.7 Cumulant Generating Function: Correlation Functions and Cumulants	33
2.7.1. Example: Cumulant of Order 4: « $X_1 X_2 X_3 X_4$ »	35
2.7.2 Significance of Cumulants	35

2.8	Gaussian and Poissonian Probability Distributions	36
2.8.1	The Gaussian Distribution	36
2.8.2	Central Limit Theorem	37
2.8.3	The Poisson Distribution	38
2.9	Limits of Sequences of Random Variables	39
2.9.1	Almost Certain Limit	40
2.9.2	Mean Square Limit (Limit in the Mean)	40
2.9.3	Stochastic Limit, or Limit in Probability	40
2.9.4	Limit in Distribution	41
2.9.5	Relationship Between Limits	41
3.	Markov Processes	
3.1	Stochastic Processes	42
3.2	Markov Process	43
3.2.1	Consistency – the Chapman-Kolmogorov Equation	43
3.2.2	Discrete State Spaces	44
3.2.3	More General Measures	44
3.3	Continuity in Stochastic Processes	45
3.3.1	Mathematical Definition of a Continuous Markov Process ..	46
3.4	Differential Chapman-Kolmogorov Equation	47
3.4.1	Derivation of the Differential Chapman-Kolmogorov Equation	48
3.4.2	Status of the Differential Chapman-Kolmogorov Equation ..	51
3.5	Interpretation of Conditions and Results	51
3.5.1	Jump Processes: The Master Equation	52
3.5.2	Diffusion Processes – the Fokker-Planck Equation	52
3.5.3	Deterministic Processes – Liouville’s Equation	53
3.5.4	General Processes	54
3.6	Equations for Time Development in Initial Time – Backward Equations	55
3.7	Stationary and Homogeneous Markov Processes	56
3.7.1	Ergodic Properties	57
3.7.2	Homogeneous Processes	60
3.7.3	Approach to a Stationary Process	61
3.7.4	Autocorrelation Function for Markov Processes	64
3.8	Examples of Markov Processes	66
3.8.1	The Wiener Process	66
3.8.2	The Random Walk in One Dimension	70
3.8.3	Poisson Process	73
3.8.4	The Ornstein-Uhlenbeck Process	75
3.8.5	Random Telegraph Process	78
4.	The Ito Calculus and Stochastic Differential Equations	
4.1	Motivation	80
4.2	Stochastic Integration	83

4.2.1	Definition of the Stochastic Integral	83
4.2.2	Example $\int_{t_0}^t W(t') dW(t')$	84
4.2.3	The Stratonovich Integral	86
4.2.4	Nonanticipating Functions	86
4.2.5	Proof that $dW(t)^2 = dt$ and $dW(t)^{2+N} = 0$	87
4.2.6	Properties of the Ito Stochastic Integral	88
4.3	Stochastic Differential Equations (SDE)	92
4.3.1	Ito Stochastic Differential Equation: Definition	93
4.3.2	Markov Property of the Solution of an Ito Stochastic Differential Equation	95
4.3.3	Change of Variables: Ito's Formula	95
4.3.4	Connection Between Fokker-Planck Equation and Stochastic Differential Equation	96
4.3.5	Multivariable Systems	97
4.3.6	Stratonovich's Stochastic Differential Equation	98
4.3.7	Dependence of Solutions on Initial Conditions and Parameters	101
4.4	Some Examples and Solutions	102
4.4.1	Coefficients Without x Dependence	102
4.4.2	Multiplicative Linear White Noise Process	103
4.4.3	Complex Oscillator with Noisy Frequency	105
4.4.4	Ornstein-Uhlenbeck Process	106
4.4.5	Conversion from Cartesian to Polar Coordinates	107
4.4.6	Multivariate Ornstein-Uhlenbeck Process	109
4.4.7	The General Single Variable Linear Equation	112
4.4.8	Multivariable Linear Equations	114
4.4.9	Time-Dependent Ornstein-Uhlenbeck Process	115
5.	The Fokker-Planck Equation	
5.1	Background	117
5.2	Fokker-Planck Equation in One Dimension	118
5.2.1	Boundary Conditions	118
5.2.2	Stationary Solutions for Homogeneous Fokker-Planck Equations	124
5.2.3	Examples of Stationary Solutions	126
5.2.4	Boundary Conditions for the Backward Fokker-Planck Equation	128
5.2.5	Eigenfunction Methods (Homogeneous Processes)	129
5.2.6	Examples	132
5.2.7	First Passage Times for Homogeneous Processes	136
5.2.8	Probability of Exit Through a Particular End of the Interval	142
5.3	Fokker-Planck Equations in Several Dimensions	143
5.3.1	Change of Variables	144
5.3.2	Boundary Conditions	146

5.3.3	Stationary Solutions: Potential Conditions	146
5.3.4	Detailed Balance	148
5.3.5	Consequences of Detailed Balance	150
5.3.6	Examples of Detailed Balance in Fokker-Planck Equations ..	155
5.3.7	Eigenfunction Methods in Many Variables – Homogeneous Processes	165
5.4	First Exit Time from a Region (Homogeneous Processes)	170
5.4.1	Solutions of Mean Exit Time Problems	171
5.4.2	Distribution of Exit Points	174
 6. Approximation Methods for Diffusion Processes		
6.1	Small Noise Perturbation Theories	177
6.2	Small Noise Expansions for Stochastic Differential Equations	180
6.2.1	Validity of the Expansion	182
6.2.2	Stationary Solutions (Homogeneous Processes)	183
6.2.3	Mean, Variance, and Time Correlation Function	184
6.2.4	Failure of Small Noise Perturbation Theories	185
6.3	Small Noise Expansion of the Fokker-Planck Equations	187
6.3.1	Equations for Moments and Autocorrelation Functions	189
6.3.2	Example	192
6.3.3	Asymptotic Method for Stationary Distributions	194
6.4	Adiabatic Elimination of Fast Variables	195
6.4.1	Abstract Formulation in Terms of Operators and Projectors	198
6.4.2	Solution Using Laplace Transform	200
6.4.3	Short-Time Behaviour	203
6.4.4	Boundary Conditions	205
6.4.5	Systematic Perturbative Analysis	206
6.5	White Noise Process as a Limit of Nonwhite Process	210
6.5.1	Generality of the Result	215
6.5.2	More General Fluctuation Equations	215
6.5.3	Time Nonhomogeneous Systems	216
6.5.4	Effect of Time Dependence in L_1	217
6.6	Adiabatic Elimination of Fast Variables: The General Case	218
6.6.1	Example: Elimination of Short-Lived Chemical Intermediates	218
6.6.2	Adiabatic Elimination in Haken's Model	223
6.6.3	Adiabatic Elimination of Fast Variables: A Nonlinear Case ..	227
6.6.4	An Example with Arbitrary Nonlinear Coupling	232
 7. Master Equations and Jump Processes		
7.1	Birth-Death Master Equations – One Variable	236
7.1.1	Stationary Solutions	236
7.1.2	Example: Chemical Reaction $X \rightleftharpoons{A}$	238
7.1.3	A Chemical Bistable System	241
7.2	Approximation of Master Equations by Fokker-Planck Equations	246
7.2.1	Jump Process Approximation of a Diffusion Process	246

7.2.2 The Kramers-Moyal Expansion	249
7.2.3 Van Kampen's System Size Expansion	250
7.2.4 Kurtz's Theorem	254
7.2.5 Critical Fluctuations	255
7.3 Boundary Conditions for Birth-Death Processes	257
7.4 Mean First Passage Times	259
7.4.1 Probability of Absorption	261
7.4.2 Comparison with Fokker-Planck Equation	261
7.5 Birth-Death Systems with Many Variables	262
7.5.1 Stationary Solutions when Detailed Balance Holds	263
7.5.2 Stationary Solutions Without Detailed Balance (Kirchoff's Solution)	266
7.5.3 System Size Expansion and Related Expansions	266
7.6 Some Examples	267
7.6.1 $X + A \rightleftharpoons 2X$	267
7.6.2 $X \xrightleftharpoons[k]{\gamma} Y \xrightleftharpoons[\gamma]{k} A$	267
7.6.3 Prey-Predator System	268
7.6.4 Generating Function Equations	273
7.7 The Poisson Representation	277
7.7.1 Kinds of Poisson Representations	282
7.7.2 Real Poisson Representations	282
7.7.3 Complex Poisson Representations	282
7.7.4 The Positive Poisson Representation	285
7.7.5 Time Correlation Functions	289
7.7.6 Trimolecular Reaction	294
7.7.7. Third-Order Noise	299
8. Spatially Distributed Systems	
8.1 Background	303
8.1.1 Functional Fokker-Planck Equations	305
8.2 Multivariate Master Equation Description	307
8.2.1 Diffusion	307
8.2.2 Continuum Form of Diffusion Master Equation	308
8.2.3 Reactions and Diffusion Combined	313
8.2.4 Poisson Representation Methods	314
8.3 Spatial and Temporal Correlation Structures	315
8.3.1 Reaction $X \xrightleftharpoons[k_2]{k_1} Y$	315
8.3.2 Reactions $B + X \xrightleftharpoons[k_3]{k_1} C, A + X \xrightarrow{k_2} 2X$	319
8.3.3 A Nonlinear Model with a Second-Order Phase Transition ..	324
8.4 Connection Between Local and Global Descriptions	328
8.4.1 Explicit Adiabatic Elimination of Inhomogeneous Modes ..	328
8.5 Phase-Space Master Equation	331
8.5.1 Treatment of Flow	331
8.5.2 Flow as a Birth-Death Process	332
8.5.3 Inclusion of Collisions – the Boltzmann Master Equation ..	336
8.5.4 Collisions and Flow Together	339

9. Bistability, Metastability, and Escape Problems	
9.1 Diffusion in a Double-Well Potential (One Variable)	342
9.1.1 Behaviour for $D = 0$	343
9.1.2 Behaviour if D is Very Small	343
9.1.3 Exit Time	345
9.1.4 Splitting Probability	345
9.1.5 Decay from an Unstable State	347
9.2 Equilibration of Populations in Each Well	348
9.2.1 Kramers' Method	349
9.2.2 Example: Reversible Denaturation of Chymotrypsinogen ..	352
9.2.3 Bistability with Birth-Death Master Equations (One Variable)	354
9.3 Bistability in Multivariable Systems	357
9.3.1 Distribution of Exit Points	357
9.3.2 Asymptotic Analysis of Mean Exit Time	362
9.3.3 Kramers' Method in Several Dimensions	363
9.3.4 Example: Brownian Motion in a Double Potential	366
10. Quantum Mechanical Markov Processes	
10.1 Quantum Mechanics of the Harmonic Oscillator	373
10.1.1 Interaction with an External Field	375
10.1.2 Properties of Coherent States	376
10.2 Density Matrix and Probabilities	380
10.2.1 Von Neumann's Equation	382
10.2.2 Glauber-Sudarshan P -Representation	382
10.2.3 Operator Correspondences	383
10.2.4 Application to the Driven Harmonic Oscillator	384
10.2.5 Quantum Characteristic Function	386
10.3 Quantum Markov Processes	388
10.3.1 Heat Bath	388
10.3.2 Correlations of Smooth Functions of Bath Operators ...	389
10.3.3 Quantum Master Equation for a System Interacting with a Heat Bath	390
10.4 Examples and Applications of Quantum Markov Processes ..	395
10.4.1 Harmonic Oscillator	395
10.4.2 The Driven Two-Level Atom	399
10.5 Time Correlation Functions in Quantum Markov Processes ..	402
10.5.1 Quantum Regression Theorem	404
10.5.2 Application to Harmonic Oscillator in the P -Representation	405
10.5.3 Time Correlations for Two-Level Atom	408
10.6 Generalised P -Representations	408
10.6.1 Definition of Generalised P -Representation	409
10.6.2 Existence Theorems	411
10.6.3 Relation to Poisson Representation	413
10.6.4 Operator Identities	414

10.7 Application of Generalised <i>P</i> -Representations to Time-Development Equations	415
10.7.1 Complex <i>P</i> -Representation	416
10.7.2 Positive <i>P</i> -Representation	416
10.7.3 Example	418
References	421
Bibliography	427
Symbol Index	431
Author Index	435
Subject Index	437