

Contents

| | | |
|----------|---|-----------|
| 1 | Brownian Motion | 1 |
| 1.1 | Definition and Construction | 1 |
| 1.2 | The Markov Property | 7 |
| 1.3 | The Right Continuous Filtration, Blumenthal's 0-1 Law | 11 |
| 1.4 | Stopping Times | 17 |
| 1.5 | The Strong Markov Property | 21 |
| 1.6 | Martingale Properties of Brownian Motion | 25 |
| 1.7 | Hitting Probabilities, Recurrence, and Transience | 27 |
| 1.8 | The Potential Kernels | 30 |
| 1.9 | Brownian Motion in a Half Space | 32 |
| 1.10 | Exit Distributions for the Sphere | 36 |
| 1.11 | Occupation Times for the Sphere | 39 |
| | Notes on Chapter 1 | 43 |
| 2 | Stochastic Integration | 44 |
| 2.1 | Integration w.r.t. Brownian Motion | 44 |
| 2.2 | Integration w.r.t. Discrete Martingales | 48 |
| 2.3 | The Basic Ingredients for Our Stochastic Integral | 50 |
| 2.4 | The Variance and Covariance of Continuous Local Martingales | 52 |
| 2.5 | Integration w.r.t. Continuous Local Martingales | 55 |
| 2.6 | The Kunita-Watanabe Inequality | 59 |
| 2.7 | Stochastic Differentials, the Associative Law | 62 |
| 2.8 | Change of Variables, Itô's Formula | 64 |
| 2.9 | Extension to Functions of Several Semimartingales | 67 |
| 2.10 | Applications of Itô's Formula | 70 |
| 2.11 | Change of Time, Lévy's Theorem | 75 |
| 2.12 | Conformal Invariance in $d \geq 2$, Kelvin's Transformations | 78 |
| 2.13 | Change of Measure, Girsanov's Formula | 82 |
| 2.14 | Martingales Adapted to Brownian Filtrations | 85 |
| | Notes on Chapter 2 | 89 |
| | A Word about the Notes | 90 |

| | | |
|----------|---|------------|
| 3 | Conditioned Brownian Motions | 91 |
| 3.1 | Warm-Up: Conditioned Random Walks | 91 |
| 3.2 | Brownian Motion Conditioned to Exit $H = R^{d-1} \times (0, \infty)$ at 0 | 94 |
| 3.3 | Other Conditioned Processes in H | 97 |
| 3.4 | Inversion in $d \geq 3$, B_t Conditioned to Converge to 0 as $t \rightarrow \infty$ | 100 |
| 3.5 | A Zero-One Law for Conditioned Processes | 102 |
| 4 | Boundary Limits of Harmonic Functions | 105 |
| 4.1 | Probabilistic Analogues of the Theorems of Privalov and Spencer | 105 |
| 4.2 | Probability Is Less Stringent than Analysis | 108 |
| 4.3 | Equivalence of Brownian and Nontangential Convergence in $d = 2$ | 113 |
| 4.4 | Burkholder and Gundy's Counterexample ($d = 3$) | 116 |
| 4.5 | With a Little Help from Analysis, Probability Works in $d \geq 3$: Brossard's Proof of Calderon's Theorem | 119 |
| 5 | Complex Brownian Motion and Analytic Functions | 123 |
| 5.1 | Conformal Invariance, Applications to Brownian Motion | 123 |
| 5.2 | Nontangential Convergence in D | 126 |
| 5.3 | Boundary Limits of Functions in the Nevanlinna Class N | 128 |
| 5.4 | Two Special Properties of Boundary Limits of Analytic Functions | 132 |
| 5.5 | Winding of Brownian Motion in $C - \{0\}$ (Spitzer's Theorem) | 134 |
| 5.6 | Tangling of Brownian Motion in $C - \{-1, 1\}$ (Picard's Theorem) | 139 |
| 6 | Hardy Spaces and Related Spaces of Martingales | 144 |
| 6.1 | Definition of H^p , an Important Example | 144 |
| 6.2 | First Definition of \mathcal{M}^p , Differences Between $p > 1$ and $p = 1$ | 146 |
| 6.3 | A Second Definition of \mathcal{M}^p | 152 |
| 6.4 | Equivalence of H^p to a Subspace of \mathcal{M}^p | 155 |
| 6.5 | Boundary Limits and Representation of Functions in H^p | 158 |
| 6.6 | Martingale Transforms | 162 |
| 6.7 | Janson's Characterization of \mathcal{M}^1 | 166 |
| 6.8 | Inequalities for Conjugate Harmonic Functions | 170 |
| 6.9 | Conjugate Functions of Indicators and Singular Measures | 180 |
| 7 | H^1 and BMO, \mathcal{M}^1 and \mathcal{BMO} | 184 |
| 7.1 | The Duality Theorem for \mathcal{M}^1 | 184 |
| 7.2 | A Second Proof of $(\mathcal{M}^1)^* = \mathcal{BMO}$ | 188 |
| 7.3 | Equivalence of BMO to a Subspace of \mathcal{BMO} | 192 |
| 7.4 | The Duality Theorem for H^1 , Fefferman-Stein Decomposition | 199 |

7.5 Examples of Martingales in \mathcal{BMO} 205
 7.6 The John-Nirenberg Inequality 208
 7.7 The Garnett-Jones Theorem 211
 7.8 A Disappointing Look at $(\mathcal{M}^p)^*$ When $p < 1$ 215

8 PDE's That Can Be Solved by Running a Brownian Motion 219

A Parabolic Equations 219
 8.1 The Heat Equation 220
 8.2 The Inhomogeneous Equation 223
 8.3 The Feynman-Kac Formula 229
 8.4 The Cameron-Martin Transformation 234
B Elliptic Equations 245
 8.5 The Dirichlet Problem 246
 8.6 Poisson's Equation 251
 8.7 The Schrödinger Equation 255
 8.8 Eigenvalues of $\Delta + c$ 263

9 Stochastic Differential Equations 271

9.1 PDE's That Can Be Solved by Running an SDE 271
 9.2 Existence of Solutions to SDE's with Continuous Coefficients 274
 9.3 Uniqueness of Solutions to SDE's with Lipschitz Coefficients 278
 9.4 Some Examples 283
 9.5 Solutions Weak and Strong, Uniqueness Questions 286
 9.6 Markov and Feller Properties 288
 9.7 Conditions for Smoothness 290
 Notes on Chapter 9 293

Appendix A Primer of Probability Theory 294

A.1 Some Differences in the Language 294
 A.2 Independence and Laws of Large Numbers 296
 A.3 Conditional Expectation 300
 A.4 Martingales 302
 A.5 Gambling Systems and the Martingale Convergence Theorem 303
 A.6 Doob's Inequality, Convergence in L^p , $p > 1$ 306
 A.7 Uniform Integrability and Convergence in L^1 307
 A.8 Optional Stopping Theorems 309

References 313

Index of Notation 325

Subject Index 327