

# Contents

Preface	vii
Suggestions for the Reader	xvii
Interdependence of the Chapters	xix
Frequently Used Notation	xxi
CHAPTER 1	
Martingales, Stopping Times, and Filtrations	1
1.1. Stochastic Processes and $\sigma$ -Fields	1
1.2. Stopping Times	6
1.3. Continuous-Time Martingales	11
A. Fundamental inequalities	12
B. Convergence results	17
C. The optional sampling theorem	19
1.4. The Doob–Meyer Decomposition	21
1.5. Continuous, Square-Integrable Martingales	30
1.6. Solutions to Selected Problems	38
1.7. Notes	45
CHAPTER 2	
Brownian Motion	47
2.1. Introduction	47
2.2. First Construction of Brownian Motion	49
A. The consistency theorem	49
B. The Kolmogorov–Čentsov theorem	53
2.3. Second Construction of Brownian Motion	56
2.4. The Space $C[0, \infty)$ , Weak Convergence, and Wiener Measure	59
A. Weak convergence	60

B. Tightness	61
C. Convergence of finite-dimensional distributions	64
D. The invariance principle and the Wiener measure	66
2.5. The Markov Property	71
A. Brownian motion in several dimensions	72
B. Markov processes and Markov families	74
C. Equivalent formulations of the Markov property	75
2.6. The Strong Markov Property and the Reflection Principle	79
A. The reflection principle	79
B. Strong Markov processes and families	81
C. The strong Markov property for Brownian motion	84
2.7. Brownian Filtrations	89
A. Right-continuity of the augmented filtration for a strong Markov process	90
B. A “universal” filtration	93
C. The Blumenthal zero-one law	94
2.8. Computations Based on Passage Times	94
A. Brownian motion and its running maximum	95
B. Brownian motion on a half-line	97
C. Brownian motion on a finite interval	97
D. Distributions involving last exit times	100
2.9. The Brownian Sample Paths	103
A. Elementary properties	103
B. The zero set and the quadratic variation	104
C. Local maxima and points of increase	106
D. Nowhere differentiability	109
E. Law of the iterated logarithm	111
F. Modulus of continuity	114
2.10. Solutions to Selected Problems	116
2.11. Notes	126
CHAPTER 3	
Stochastic Integration	128
3.1. Introduction	128
3.2. Construction of the Stochastic Integral	129
A. Simple processes and approximations	132
B. Construction and elementary properties of the integral	137
C. A characterization of the integral	141
D. Integration with respect to continuous, local martingales	145
3.3. The Change-of-Variable Formula	148
A. The Itô rule	149
B. Martingale characterization of Brownian motion	156
C. Bessel processes, questions of recurrence	158
D. Martingale moment inequalities	163
E. Supplementary exercises	167
3.4. Representations of Continuous Martingales in Terms of Brownian Motion	169
A. Continuous local martingales as stochastic integrals with respect to Brownian motion	170

B. Continuous local martingales as time-changed Brownian motions	173
C. A theorem of F. B. Knight	179
D. Brownian martingales as stochastic integrals	180
E. Brownian functionals as stochastic integrals	185
3.5. The Girsanov Theorem	190
A. The basic result	191
B. Proof and ramifications	193
C. Brownian motion with drift	196
D. The Novikov condition	198
3.6. Local Time and a Generalized Itô Rule for Brownian Motion	201
A. Definition of local time and the Tanaka formula	203
B. The Trotter existence theorem	206
C. Reflected Brownian motion and the Skorohod equation	210
D. A generalized Itô rule for convex functions	212
E. The Engelbert–Schmidt zero-one law	215
3.7. Local Time for Continuous Semimartingales	217
3.8. Solutions to Selected Problems	226
3.9. Notes	236
CHAPTER 4	
Brownian Motion and Partial Differential Equations	239
4.1. Introduction	239
4.2. Harmonic Functions and the Dirichlet Problem	240
A. The mean-value property	241
B. The Dirichlet problem	243
C. Conditions for regularity	247
D. Integral formulas of Poisson	251
E. Supplementary exercises	253
4.3. The One-Dimensional Heat Equation	254
A. The Tychonoff uniqueness theorem	255
B. Nonnegative solutions of the heat equation	256
C. Boundary crossing probabilities for Brownian motion	262
D. Mixed initial/boundary value problems	265
4.4. The Formulas of Feynman and Kac	267
A. The multidimensional formula	268
B. The one-dimensional formula	271
4.5. Solutions to selected problems	275
4.6. Notes	278
CHAPTER 5	
Stochastic Differential Equations	281
5.1. Introduction	281
5.2. Strong Solutions	284
A. Definitions	285
B. The Itô theory	286
C. Comparison results and other refinements	291
D. Approximations of stochastic differential equations	295
E. Supplementary exercises	299

5.3. Weak Solutions	300
A. Two notions of uniqueness	301
B. Weak solutions by means of the Girsanov theorem	302
C. A digression on regular conditional probabilities	306
D. Results of Yamada and Watanabe on weak and strong solutions	308
5.4. The Martingale Problem of Stroock and Varadhan	311
A. Some fundamental martingales	312
B. Weak solutions and martingale problems	314
C. Well-posedness and the strong Markov property	319
D. Questions of existence	323
E. Questions of uniqueness	325
F. Supplementary exercises	328
5.5. A Study of the One-Dimensional Case	329
A. The method of time change	330
B. The method of removal of drift	339
C. Feller's test for explosions	342
D. Supplementary exercises	351
5.6. Linear Equations	354
A. Gauss–Markov processes	355
B. Brownian bridge	358
C. The general, one-dimensional, linear equation	360
D. Supplementary exercises	361
5.7. Connections with Partial Differential Equations	363
A. The Dirichlet problem	364
B. The Cauchy problem and a Feynman–Kac representation	366
C. Supplementary exercises	369
5.8. Applications to Economics	371
A. Portfolio and consumption processes	371
B. Option pricing	376
C. Optimal consumption and investment (general theory)	379
D. Optimal consumption and investment (constant coefficients)	381
5.9. Solutions to Selected Problems	387
5.10. Notes	394
CHAPTER 6	
P. Lévy's Theory of Brownian Local Time	399
6.1. Introduction	399
6.2. Alternate Representations of Brownian Local Time	400
A. The process of passage times	400
B. Poisson random measures	403
C. Subordinators	405
D. The process of passage times revisited	411
E. The excursion and downcrossing representations of local time	414
6.3. Two Independent Reflected Brownian Motions	418
A. The positive and negative parts of a Brownian motion	418
B. The first formula of D. Williams	421
C. The joint density of $(W(t), L(t), \Gamma_+(t))$	423

6.4. Elastic Brownian Motion	425
A. The Feynman–Kac formulas for elastic Brownian motion	426
B. The Ray–Knight description of local time	430
C. The second formula of D. Williams	434
6.5. An Application: Transition Probabilities of Brownian Motion with Two-Valued Drift	437
6.6. Solutions to Selected Problems	442
6.7. Notes	445
Bibliography	447
Index	459