Contents

		page
Prerequisites		1
Chapter 1. The standard Brownian motion		5
1.1.	The standard random walk	5
1.2.	Passage times for the standard random walk	7
1.3.	Hinčin's proof of the de Moivre-Laplace limit theorem	10
1.4.	The standard Brownian motion	12
1.5.	P. Lévy's construction	19
1.6.	Strict Markov character	22
1.7.	Passage times for the standard Brownian motion	25
	Note 1: Homogeneous differential processes with increasing paths	31
1.8.	Kolmogorov's test and the law of the iterated logarithm	33
1.9.	P. Lévy's Hölder condition	36
1.10.	Approximating the Brownian motion by a random walk	38
Chapter 2. Brownian local times		40
2.1.	The reflecting Brownian motion	40
2.2.	P. Lévy's local time	42
2.3.	Elastic Brownian motion	45
2.4.	t^+ and down-crossings	48
2.5.	t^+ as Hausdorff-Besicovitch 1/2-dimensional measure	50
	Note 1: Submartingales	52
	Note 2: Hausdorff measure and dimension	53
2.6.	Kac's formula for Brownian functionals	54
2.7.	Bessel processes	59
2.8.	Standard Brownian local time	63
2.9.	Brownian excursions	75
2.10.	Application of the Bessel process to Brownian excursions	79
2.11.	A time substitution	81
Chapter 3. The general 1-dimensional diffusion		83
3.1.	Definition	83
3.2.	Markov times	86
3.3.	Matching numbers	89
3.4.	Singular points	91
3.5.	Decomposing the general diffusion into simple pieces	92
3.6.	Green operators and the space D	94
3.7.	Generators	98
3.8.	Generators continued	100
3.9.	Stopped diffusion	102

page

Chapter 4. Generators 105

4.1. A general view 105
4.2. $\mathfrak{G}$ as local differential operator: conservative non-singular case . . 111
4.3. $\mathfrak{G}$ as local differential operator: general non-singular case 116
4.4. A second proof 119
4.5. $\mathfrak{G}$ at an isolated singular point 125
4.6. Solving $\mathfrak{G}^{\bullet} u = \alpha u$ 128
4.7. $\mathfrak{G}$ as global differential operator: non-singular case 135
4.8. $\mathfrak{G}$ on the shunts 136
4.9. $\mathfrak{G}$ as global differential operator: singular case 142
4.10. Passage times 144
Note 1: Differential processes with increasing paths 146
4.11. Eigen-differential expansions for Green functions and transition densities 149
4.12. Kolmogorov's test 161

Chapter 5. Time changes and killing 164

5.1. Construction of sample paths: a general view 164
5.2. Time changes: $Q = R^1$ 167
5.3. Time changes: $Q = [0, +\infty)$ 171
5.4. Local times 174
5.5. Subordination and chain rule 176
5.6. Killing times 179
5.7. Feller's Brownian motions 186
5.8. Ikeda's example 188
5.9. Time substitutions must come from local time integrals 190
5.10. Shunts 191
5.11. Shunts with killing 196
5.12. Creation of mass 200
5.13. A parabolic equation 201
5.14. Explosions 206
5.15. A non-linear parabolic equation 209

Chapter 6. Local and inverse local times 212

6.1. Local and inverse local times 212
6.2. Lévy measures 214
6.3. $\mathfrak{t}$ and the intervals of $[0, +\infty) - \mathfrak{Z}$ 218
6.4. A counter example: $\mathfrak{t}$ and the intervals of $[0, +\infty) - \mathfrak{Z}$ 220
6.5a $\mathfrak{t}$ and downcrossings 222
6.5b $\mathfrak{t}$ as Hausdorff measure 223
6.5c $\mathfrak{t}$ as diffusion 223
6.5d Excursions 223
6.6. Dimension numbers 224
6.7. Comparison tests 225
Note 1: Dimension numbers and fractional dimensional capacities 227
6.8. An individual ergodic theorem 228

Chapter 7. Brownian motion in several dimensions 232

7.1. Diffusion in several dimensions 232
7.2. The standard Brownian motion in several dimensions 233
7.3. Wandering out to ∞ 236

page

7.4. GREENian domains and GREEN functions 237
7.5. Excessive functions 243
7.6. Application to the spectrum of $\Delta/2$ 245
7.7. Potentials and hitting probabilities 247
7.8. NEWTONian capacities 250
7.9. GAUSS's quadratic form 253
7.10. WIENER's test 255
7.11. Applications of WIENER's test 257
7.12. DIRICHLET problem 261
7.13. NEUMANN problem 264
7.14. Space-time BROWNian motion 266
7.15. Spherical BROWNian motion and skew products 269
7.16. Spinning 274
7.17. An individual ergodic theorem for the standard 2-dimensional BROWNian motion 277
7.18. Covering BROWNian motions 279
7.19. Diffusions with BROWNian hitting probabilities 283
7.20. Right-continuous paths 286
7.21. RIESZ potentials 288

Chapter 8. A general view of diffusion in several dimensions . . . 291

8.1. Similar diffusions 291
8.2. $\mathfrak{G}$ as differential operator 293
8.3. Time substitutions 295
8.4. Potentials 296
8.5. Boundaries 299
8.6. Elliptic operators 302
8.7. FELLER's little boundary and tail algebras 303

Bibliography 306

List of notations 313

Index 315