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$$\underbrace{\sum_{q=1}^{q=m}\sum_{r=1}^{r=n_q}\frac{w_{rq}(x_{rq}-\mu)^2}{\sigma^2}}_{\chi_0^2} =$$

$$\underbrace{\sum_{q=1}^{q=m}\sum_{r=1}^{r=n_q}\frac{w_{rq}(x_{rq}-\bar{x}_q)^2}{\sigma^2}}_{\chi_1^2} + \underbrace{\sum_{q=1}^{q=m}\sum_{r=1}^{r=n_q}\frac{w_{rq}(x_{rq}-\bar{x})^2}{\sigma^2}}_{\chi_2^2} + \underbrace{\sum_{q=1}^{q=m}\sum_{r=1}^{r=n_q}\frac{w_{rq}(x_{rq}-\mu)^2}{\sigma^2}}_{\chi_3^2}$$

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