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About these notes. Acknowledgements

Suggestions on how to use them (*strongly recommended reading!*)

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