Table of Contents

Chapter 0. Preliminaries 1

§1. Basic Notation 1
§2. Monotone Class Theorem 2
§3. Completion 3
§4. Functions of Finite Variation and Stieltjes Integrals 4
§5. Weak Convergence in Metric Spaces 9
§6. Gaussian and Other Random Variables 11

Chapter I. Introduction 15

§1. Examples of Stochastic Processes. Brownian Motion 15
§2. Local Properties of Brownian Paths 26
§3. Canonical Processes and Gaussian Processes 33
§4. Filtrations and Stopping Times 41
Notes and Comments 48

Chapter II. Martingales 51

§1. Definitions, Maximal Inequalities and Applications 51
§2. Convergence and Regularization Theorems 60
§3. Optional Stopping Theorem 68
Notes and Comments 77

Chapter III. Markov Processes 79

§1. Basic Definitions 79
§2. Feller Processes 88
§3. Strong Markov Property 102
§4. Summary of Results on Lévy Processes 114
Notes and Comments 117

Chapter IV. Stochastic Integration 119

§1. Quadratic Variations 119
§2. Stochastic Integrals 137

§3. Itô's Formula and First Applications 146
§4. Burkholder-Davis-Gundy Inequalities 160
§5. Predictable Processes 171
Notes and Comments 176

Chapter V. Representation of Martingales 179

§1. Continuous Martingales as Time-changed Brownian Motions 179
§2. Conformal Martingales and Planar Brownian Motion 189
§3. Brownian Martingales 198
§4. Integral Representations 209
Notes and Comments 216

Chapter VI. Local Times 221

§1. Definition and First Properties 221
§2. The Local Time of Brownian Motion 238
§3. The Three-Dimensional Bessel Process 251
§4. First Order Calculus 260
§5. The Skorokhod Stopping Problem 269
Notes and Comments 277

Chapter VII. Generators and Time Reversal 281

§1. Infinitesimal Generators 281
§2. Diffusions and Itô Processes 293
§3. Linear Continuous Markov Processes 300
§4. Time Reversal and Applications 312
Notes and Comments 322

Chapter VIII. Girsanov's Theorem and First Applications 325

§1. Girsanov's Theorem 325
§2. Application of Girsanov's Theorem to the Study of Wiener's Space 338
§3. Functionals and Transformations of Diffusion Processes 349
Notes and Comments 362

Chapter IX. Stochastic Differential Equations 365

§1. Formal Definitions and Uniqueness 365
§2. Existence and Uniqueness in the Case of Lipschitz Coefficients 375
§3. The Case of Hölder Coefficients in Dimension One 388
Notes and Comments 399

Chapter X. Additive Functionals of Brownian Motion 401

§1. General Definitions 401

§2. Representation Theorem for Additive Functionals of Linear Brownian Motion 409
§3. Ergodic Theorems for Additive Functionals 422
§4. Asymptotic Results for the Planar Brownian Motion 430
Notes and Comments 436

Chapter XI. Bessel Processes and Ray-Knight Theorems 439

§1. Bessel Processes 439
§2. Ray-Knight Theorems 454
§3. Bessel Bridges 463
Notes and Comments 469

Chapter XII. Excursions 471

§1. Prerequisites on Poisson Point Processes 471
§2. The Excursion Process of Brownian Motion 480
§3. Excursions Straddling a Given Time 488
§4. Descriptions of Itô's Measure and Applications 493
Notes and Comments 511

Chapter XIII. Limit Theorems in Distribution 515

§1. Convergence in Distribution 515
§2. Asymptotic Behavior of Additive Functionals of Brownian Motion 522
§3. Asymptotic Properties of Planar Brownian Motion 531
Notes and Comments 541

Appendix 543

§1. Gronwall's Lemma 543
§2. Distributions 543
§3. Convex Functions 544
§4. Hausdorff Measures and Dimension 547
§5. Ergodic Theory 548
§6. Probabilities on Function Spaces 548
§7. Bessel Functions 549
§8. Sturm-Liouville Equation 550

Bibliography 553

Index of Notation 591

Index of Terms 595

Catalogue 601