

CONTENTS

1. Introduction	1
1.1 Factors, levels, cells and effects	2
1.2 Balanced and unbalanced data	4
a. <i>Balanced data</i> , 4	
b. <i>Special cases of unbalanced data</i> , 4	
-i. <i>Planned unbalancedness</i> , 4	
-ii. <i>Estimating missing observations</i> , 6	
c. <i>Unbalanced data</i> , 6	
1.3 Fixed effects and random effects	7
a. <i>Fixed effects models</i> , 7	
<i>Example 1 (Tomato varieties)</i> , 7	
<i>Example 2 (Medications)</i> , 8	
<i>Example 3 (Soils and fertilizers)</i> , 9	
b. <i>Random effects models</i> , 9	
<i>Example 4 (Clinics)</i> , 9	
<i>Example 5 (Dairy bulls)</i> , 12	
<i>Example 6 (Ball bearings and calipers)</i> , 13	
c. <i>Mixed models</i> , 14	
<i>Example 7 (Medications and clinics)</i> , 14	
<i>Example 8 (Varieties and gardens)</i> , 14	
1.4 Fixed or random?	15
<i>Example 9 (Mice and technicians)</i> , 15	
1.5 Finite populations	16
1.6 Summary	17
a. <i>Characteristics of the fixed effects model and the random effects model for the 1-way classification</i> , 17	
b. <i>Examples</i> , 17	
c. <i>Fixed or random</i> , 18	
2. History and Comment	19
2.1 Analysis of variance	19

2.2	Early years: 1861–1949	22
	<i>a. Sources</i> , 22	
	<i>b. Pre-1900</i> , 23	
	<i>c. 1900–1939</i> , 25	
	-i. <i>R. A. Fisher</i> , 25	
	-ii. <i>L. C. Tippett</i> , 27	
	-iii. <i>The late 1930s</i> , 29	
	-iv. <i>Unbalanced data</i> , 31	
	<i>d. The 1940s</i> , 32	
2.3	Great strides: 1950–1969	33
	<i>a. The Henderson methods</i> , 34	
	<i>b. ANOVA estimation, in general</i> , 35	
	-i. <i>Negative estimates</i> , 35	
	-ii. <i>Unbiasedness</i> , 36	
	-iii. <i>Best unbiasedness</i> , 37	
	-iv. <i>Minimal sufficient statistics</i> , 38	
	-v. <i>Lack of uniqueness</i> , 38	
2.4	Into the 1970s and beyond	40
	<i>a. Maximum likelihood (ML)</i> , 40	
	<i>b. Restricted maximum likelihood (REML)</i> , 41	
	<i>c. Minimum norm estimation</i> , 41	
	<i>d. The dispersion-mean model</i> , 42	
	<i>e. Bayes estimation</i> , 42	
	<i>f. The recent decade</i> , 43	
3.	The 1-Way Classification	44
3.1	The model	44
	<i>a. The model equation</i> , 44	
	<i>b. First moments</i> , 45	
	<i>c. Second moments</i> , 45	
3.2	Matrix formulation of the model	47
	<i>a. Example 1</i> , 47	
	<i>b. The general case</i> , 48	
	<i>c. Dispersion matrices</i> , 49	
	-i. <i>The traditional random model</i> , 49	
	-ii. <i>Other alternatives</i> , 50	
	<i>d. Unbalanced data</i> , 51	
	-i. <i>Example 2</i> , 51	
	-ii. <i>The general case</i> , 52	
	-iii. <i>Dispersion matrix</i> , 52	
3.3	Estimating the mean	52
3.4	Predicting random effects	54
3.5	ANOVA estimation—balanced data	57
	<i>a. Expected sums of squares</i> , 57	
	-i. <i>A direct derivation</i> , 58	
	-ii. <i>Using the matrix formulation</i> , 58	
	<i>b. ANOVA estimators</i> , 59	

	c. <i>Negative estimates</i> , 60	
	d. <i>Normality assumptions</i> , 62	
	-i. χ^2 -distributions of sums of squares, 62	
	-ii. <i>Independence of sums of squares</i> , 63	
	-iii. <i>Sampling variances of estimators</i> , 63	
	-iv. <i>An F-statistic to test H: $\sigma_\alpha^2 = 0$</i> , 64	
	-v. <i>Confidence intervals</i> , 65	
	-vi. <i>Probability of a negative estimate</i> , 66	
	-vii. <i>Distribution of estimators</i> , 69	
3.6	ANOVA estimation—unbalanced data	69
	a. <i>Expected sums of squares</i> , 69	
	-i. <i>A direct derivation</i> , 69	
	-ii. <i>Using the matrix formulation</i> , 70	
	b. <i>ANOVA estimators</i> , 71	
	c. <i>Negative estimates</i> , 72	
	d. <i>Normality assumptions</i> , 73	
	-i. χ^2 -distributions of sums of squares, 73	
	-ii. <i>Independence of sums of squares</i> , 73	
	-iii. <i>Sampling variances of estimators</i> , 74	
	-iv. <i>The effect of unbalancedness on sampling variances</i> , 75	
	-v. <i>F-statistics</i> , 76	
	-vi. <i>Confidence intervals</i> , 76	
3.7	Maximum likelihood estimation (MLE)	78
	a. <i>Balanced data</i> , 79	
	-i. <i>Likelihood</i> , 79	
	-ii. <i>ML equations and their solutions</i> , 80	
	-iii. <i>ML estimators</i> , 81	
	-iv. <i>Expected values and bias</i> , 84	
	-v. <i>Sampling variances</i> , 85	
	b. <i>Unbalanced data</i> , 86	
	-i. <i>Likelihood</i> , 86	
	-ii. <i>ML equations and their solutions</i> , 87	
	-iii. <i>ML estimators</i> , 88	
	-iv. <i>Bias</i> , 88	
	-v. <i>Sampling variances</i> , 88	
3.8	Restricted maximum likelihood estimation (REML)	90
	a. <i>Balanced data</i> , 91	
	-i. <i>Likelihood</i> , 91	
	-ii. <i>REML equations and their solutions</i> , 91	
	-iii. <i>REML estimators</i> , 92	
	-iv. <i>Comparison with ANOVA and ML</i> , 92	
	-v. <i>Bias</i> , 93	
	-vi. <i>Sampling variances</i> , 93	
	b. <i>Unbalanced data</i> , 93	
3.9	Bayes estimation	94
	a. <i>A simple sample</i> , 94	
	b. <i>The 1-way classification, random model</i> , 97	
	c. <i>Balanced data</i> , 99	

3.10	A summary	103
	<i>a. Balanced data</i> , 103	
	<i>b. Unbalanced data</i> , 106	
3.11	Exercises	108
4.	Balanced Data	112
4.1	Establishing analysis of variance tables	113
	<i>a. Factors and levels</i> , 113	
	<i>b. Lines in the analysis of variance tables</i> , 113	
	<i>c. Interactions</i> , 113	
	<i>d. Degrees of freedom</i> , 114	
	<i>e. Sums of squares</i> , 114	
	<i>f. Calculating sums of squares</i> , 115	
4.2	Expected mean squares, $E(MS)$	116
4.3	The 2-way crossed classification	118
	<i>a. Introduction</i> , 118	
	<i>b. Analysis of variance table</i> , 118	
	<i>c. Expected mean squares</i> , 119	
	-i. <i>The fixed effects model</i> , 121	
	-ii. <i>The random effects model</i> , 122	
	-iii. <i>The mixed model</i> , 122	
	-iv. <i>A mixed model with Σ-restrictions</i> , 123	
	<i>d. ANOVA estimators of variance components</i> , 127	
4.4	ANOVA estimation	128
4.5	Normality assumptions	131
	<i>a. Distribution of mean squares</i> , 131	
	<i>b. Distribution of estimators</i> , 132	
	<i>c. Tests of hypotheses</i> , 133	
	<i>d. Confidence intervals</i> , 135	
	<i>e. Probability of a negative estimate</i> , 137	
	<i>f. Sampling variances of estimators</i> , 137	
4.6	A matrix formulation of mixed models	138
	<i>a. The general mixed model</i> , 138	
	<i>b. The 2-way crossed classification</i> , 140	
	-i. <i>Model equation</i> , 140	
	-ii. <i>Random or mixed?</i> , 142	
	-iii. <i>Dispersion matrix</i> , 142	
	<i>c. The 2-way nested classification</i> , 142	
	<i>d. Interaction or nested factor?</i> , 143	
	<i>e. The general case</i> , 143	
	-i. <i>Model equation</i> , 144	
	-ii. <i>Dispersion matrix</i> , 144	
4.7	Maximum likelihood estimation (ML)	146
	<i>a. Estimating the mean in random models</i> , 146	
	<i>b. Four models with closed form estimators</i> , 147	
	-i. <i>The 1-way random model</i> , 147	
	-ii. <i>The 2-way nested random model</i> , 148	
	-iii. <i>The 2-way crossed, with interaction, mixed model</i> , 149	

	-iv. <i>The 2-way crossed, no interaction, mixed model</i> , 150	
	c. <i>Unbiasedness</i> , 151	
	d. <i>The 2-way crossed classification, random model</i> , 151	
	-i. <i>With interaction</i> , 152	
	-ii. <i>No interaction</i> , 153	
	e. <i>Existence of explicit solutions</i> , 153	
	f. <i>Asymptotic sampling variances for the 2-way crossed classification</i> , 154	
	-i. <i>The 2-way crossed classification, with interaction, random model</i> , 155	
	-ii. <i>The 2-way crossed classification, no interaction, random model</i> , 155	
	-iii. <i>The 2-way crossed classification, with interaction, mixed model</i> , 156	
	-iv. <i>The 2-way crossed classification, no interaction, mixed model</i> , 156	
	g. <i>Asymptotic sampling variances for two other models</i> , 157	
	-i. <i>The 2-way nested classification, random model</i> , 157	
	-ii. <i>The 1-way classification, random model</i> , 158	
	h. <i>Locating results</i> , 158	
4.8	Restricted maximum likelihood	158
4.9	Estimating fixed effects in mixed models	159
4.10	Summary	161
4.11	Exercises	163
5.	Analysis of Variance Estimation for Unbalanced Data	168
5.1	Model formulation	169
	a. <i>Data</i> , 169	
	b. <i>A general model</i> , 170	
	-i. <i>Example 1—the 2-way crossed classification random model</i> , 170	
	-ii. <i>Dispersion matrix</i> , 171	
	-iii. <i>Example 1 (continued)</i> , 172	
5.2	ANOVA estimation	172
	a. <i>Example 2—the 1-way random model, balanced data</i> , 172	
	b. <i>Estimation</i> , 172	
	-i. <i>The general case</i> , 172	
	-ii. <i>Example 2 (continued)</i> , 173	
	-iii. <i>Example 1 (continued)</i> , 173	
	c. <i>Unbiasedness</i> , 175	
	d. <i>Sampling variances</i> , 176	
	-i. <i>A general result</i> , 176	
	-ii. <i>Example 2 (continued)</i> , 177	
	-iii. <i>A direct approach</i> , 178	
	e. <i>Unbiased estimation of sampling variances</i> , 179	
5.3	Henderson's Method I	181
	a. <i>The quadratic forms</i> , 181	
	b. <i>Estimation</i> , 183	

	c. <i>Negative estimates</i> , 184	
	d. <i>Sampling variances</i> , 184	
	e. <i>A general coefficient for Method I</i> , 186	
	f. <i>Synthesis</i> , 187	
	g. <i>Mixed models</i> , 188	
	h. <i>Merits and demerits</i> , 189	
	i. <i>A numerical example</i> , 189	
5.4	Henderson's Method II	190
	a. <i>Estimating the fixed effects</i> , 191	
	b. <i>Calculation</i> , 192	
	c. <i>Verification</i> , 194	
	-i. <i>The matrix \mathbf{XLZ} is null</i> , 194	
	-ii. <i>Row sums of \mathbf{XL} are all the same</i> , 195	
	-iii. <i>All rows of $\mathbf{X} - \mathbf{XLX}$ are the same</i> , 195	
	d. <i>Invariance</i> , 196	
	-i. <i>Rank properties</i> , 196	
	-ii. <i>An estimable function</i> , 197	
	-iii. <i>Two solutions</i> , 198	
	-iv. <i>The quadratic forms</i> , 198	
	e. <i>Coefficients of σ_e^2</i> , 198	
	f. <i>No fixed-by-random interactions</i> , 199	
	g. <i>Merits and demerits</i> , 201	
5.5	Henderson's Method III	202
	a. <i>Borrowing from fixed effects models</i> , 202	
	-i. <i>Reductions in sum of squares</i> , 202	
	-ii. <i>Expected sums of squares</i> , 203	
	b. <i>Mixed models</i> , 204	
	c. <i>A general result</i> , 206	
	d. <i>Sampling variances</i> , 207	
	e. <i>Merits and demerits</i> , 208	
5.6	Method III applied to the 2-way crossed classification	208
	a. <i>No interaction, random model</i> , 209	
	-i. <i>One set of sums of squares</i> , 209	
	-ii. <i>Three sets of sums of squares</i> , 210	
	-iii. <i>Calculation</i> , 213	
	-iv. <i>Sampling variances</i> , 213	
	b. <i>No interaction, mixed model</i> , 213	
	c. <i>With interaction, random model</i> , 213	
	-i. <i>One set of sums of squares</i> , 214	
	-ii. <i>Three sets of sums of squares</i> , 215	
	-iii. <i>Calculation</i> , 217	
	-iv. <i>Sampling variances</i> , 217	
	d. <i>With interaction, mixed models</i> , 218	
5.7	Nested models	218
5.8	Other forms of ANOVA estimation	219
	a. <i>Unweighted means method: all cells filled</i> , 219	
	b. <i>Weighted squares of means: all cells filled</i> , 220	
5.9	Comparing different forms of ANOVA estimation	221
5.10	Estimating fixed effects in mixed models	225

5.11	Summary	226
5.12	Exercises	227
6.	Maximum Likelihood (ML) and Restricted Maximum Likelihood (REML)	232
6.1	The model and likelihood function	233
6.2	The ML estimation equations	234
	<i>a. A direct derivation, 234</i>	
	<i>b. An alternative form, 236</i>	
	<i>c. The Hartley–Rao form, 237</i>	
6.3	Asymptotic dispersion matrices for ML estimators	238
	<i>a. For variance components, 238</i>	
	<i>b. For ratios of components, 240</i>	
	<i>c. Maximum?, 241</i>	
6.4	Some remarks on computing	242
6.5	ML results for 2-way crossed classification, balanced data	243
	<i>a. 2-way crossed, random model, with interaction, 243</i>	
	<i>-i. Notation, 243</i>	
	<i>-ii. Inverse of V, 244</i>	
	<i>-iii. The estimation equations, 245</i>	
	<i>-iv. Information matrix, 247</i>	
	<i>b. 2-way crossed, random model, no interaction, 249</i>	
6.6	Restricted maximum likelihood (REML)	249
	<i>a. Linear combinations of observations, 250</i>	
	<i>b. The REML equations, 251</i>	
	<i>c. An alternative form, 251</i>	
	<i>d. Invariance to choice of error contrasts, 252</i>	
	<i>e. The information matrix, 252</i>	
	<i>f. Balanced data, 253</i>	
	<i>g. Using cell means models for the fixed effects, 253</i>	
6.7	Estimating fixed effects in mixed models	254
	<i>a. ML, 254</i>	
	<i>b. REML, 254</i>	
6.8	ML or REML?	254
6.9	Summary	255
6.10	Exercises	256
7.	Prediction of Random Variables	258
7.1	Introduction	258
7.2	Best prediction (BP)	261
	<i>a. The best predictor, 261</i>	
	<i>b. Mean and variance properties, 262</i>	
	<i>c. Two properties of the best predictor of a scalar, 263</i>	
	<i>-i. Maximizing a correlation, 264</i>	
	<i>-ii. Maximizing the mean of a selected proportion, 264</i>	
	<i>d. Normality, 265</i>	

7.3	Best linear prediction (BLP)	265
	a. BLP(\mathbf{u}), 265	
	b. Example, 266	
	c. Derivation, 267	
	d. Ranking, 268	
7.4	Mixed model prediction (BLUP)	269
	a. Combining fixed and random effects, 269	
	b. Example, 270	
	c. Derivation of BLUP, 271	
	d. Variances and covariances, 272	
	e. Normality, 273	
7.5	Other derivations of BLUP	273
	a. A two-stage derivation, 273	
	b. A direct derivation assuming linearity, 274	
	c. Partitioning \mathbf{y} into two parts, 274	
	d. A Bayes estimator, 275	
7.6	Henderson's mixed model equations (MME)	275
	a. Derivation, 275	
	b. Solutions, 276	
	c. Calculations for ML estimation, 278	
	-i. The estimation equations, 278	
	-ii. The information matrix, 280	
	d. Calculations for REML estimation, 282	
	-i. The estimation equations, 282	
	-ii. The information matrix, 284	
	e. Iterative procedures summarized, 284	
	-i. Adapting the MME, 284	
	-ii. Using the MME, 284	
	-iii. Iterating for ML, 285	
	-iv. Iterating for REML, 285	
	f. A summary, 286	
7.7	Summary	286
7.8	Exercises	287
8.	Computing ML and REML estimates	290
8.1	Introduction	290
8.2	Iterative methods based on derivatives	292
	a. The basis of the methods, 292	
	b. The Newton–Raphson and Marquardt methods, 293	
	c. Method of scoring, 295	
	d. Quasi-Newton methods, 295	
	e. Obtaining starting values, 295	
	f. Termination rules, 296	
	g. Incorporation of non-negativity constraints, 296	
	h. Easing the computational burden, 296	
8.3	The EM algorithm	297
	a. A general formulation, 297	
	b. Distributional derivations needed for the EM algorithm, 298	

c.	<i>EM algorithm for ML estimation (Version 1)</i> , 300	
d.	<i>EM algorithm for ML estimation (Version 2)</i> , 301	
e.	<i>Equivalence of the EM algorithm to the ML equations</i> , 302	
f.	<i>EM algorithm for REML estimation</i> , 302	
g.	<i>A Bayesian justification for REML</i> , 303	
h.	<i>Non-zero correlations among the u_is</i> , 304	
8.4	General methods that converge rapidly for balanced data	304
8.5	Pooling estimators from subsets of a large data set	305
8.6	Example: the 1-way random model	307
a.	<i>The EM algorithm (Version 1)</i> , 308	
b.	<i>The method of scoring algorithm</i> , 310	
8.7	Discussion	311
a.	<i>Computing packages</i> , 311	
b.	<i>Evaluation of algorithms</i> , 312	
8.8	Summary	313
8.9	Exercises	314
9.	Hierarchical Models and Bayesian Estimation	315
9.1	Basic principles	315
a.	<i>Introduction</i> , 315	
b.	<i>Simple examples</i> , 316	
c.	<i>The mixed model hierarchy</i> , 317	
d.	<i>The normal hierarchy</i> , 318	
e.	<i>Point estimator of variance or variance of point estimator</i> , 319	
9.2	Variance estimation in the normal hierarchy	321
a.	<i>Formal hierarchical estimation</i> , 321	
b.	<i>Likelihood methods</i> , 321	
c.	<i>Empirical Bayes estimation</i> , 325	
-i.	<i>General strategies</i> , 325	
-ii.	<i>Estimation</i> , 325	
-iii.	<i>Connections with likelihood</i> , 326	
9.3	Estimation of effects	327
a.	<i>Hierarchical estimation</i> , 327	
-i.	<i>Estimation of β</i> , 328	
-ii.	<i>Estimation of u</i> , 330	
b.	<i>An alternative derivation</i> , 331	
-i.	<i>Exploiting the multivariate normal structure</i> , 331	
-ii.	<i>Relationship to BLUP</i> , 333	
c.	<i>The 1-way classification, random model</i> , 333	
-i.	<i>Estimation of μ</i> , 334	
-ii.	<i>Estimation of α</i> , 335	
d.	<i>Empirical Bayes estimation</i> , 337	
-i.	<i>The 1-way classification</i> , 337	
-ii.	<i>Cautions</i> , 338	
-iii.	<i>Variance approximations</i> , 341	
9.4	Other types of hierarchies	343
a.	<i>A beta-binomial model</i> , 344	
b.	<i>A generalized linear model</i> , 347	

9.5	Practical considerations in hierarchical modeling	349
	<i>a. Computational problems</i> , 349	
	<i>b. Hierarchical EM</i> , 350	
9.6	Philosophical considerations in hierarchical modeling	352
9.7	Summary	355
9.8	Exercises	359
10.	Binary and Discrete Data	367
10.1	Introduction	367
10.2	ANOVA methods	369
10.3	Beta–binomial models	369
	<i>a. Introduction</i> , 369	
	<i>b. Model specification</i> , 370	
	<i>c. Likelihood</i> , 371	
	<i>d. Discussion</i> , 371	
10.4	Logit–normal models	372
10.5	Probit–normal models	373
	<i>a. Introduction</i> , 373	
	<i>b. An example</i> , 373	
10.6	Discussion	374
10.7	Summary	375
10.8	Exercises	376
11.	Other Procedures	378
11.1	Estimating components of covariance	378
	<i>a. Easy ANOVA estimation for certain models</i> , 379	
	<i>b. Examples of covariance components models</i> , 380	
	-i. <i>Covariances between effects of the same random factor</i> , 381	
	-ii. <i>Covariances between effects of different random factors</i> , 381	
	-iii. <i>Covariances between error terms</i> , 382	
	<i>c. Combining variables into a single vector</i> , 382	
	<i>d. Genetic covariances</i> , 383	
	<i>e. Maximum likelihood (ML) estimation</i> , 383	
	-i. <i>Estimation equations</i> , 383	
	-ii. <i>Large-sample dispersion matrix</i> , 385	
	<i>f. Restricted maximum likelihood (REML) estimation</i> , 386	
	-i. <i>Estimation equations</i> , 386	
	-ii. <i>Large-sample dispersion matrix</i> , 387	
11.2	Modeling variance components as covariances	387
	<i>a. All-cells-filled data</i> , 388	
	<i>b. Balanced data</i> , 389	
	<i>c. Diagnostic opportunities</i> , 389	
	<i>d. Some-cells-empty data</i> , 390	
11.3	Criteria-based procedures	391
	<i>a. Three criteria</i> , 392	
	-i. <i>Unbiasedness</i> , 392	
	-ii. <i>Translation invariance</i> , 392	

	-iii. Minimum variance, 393	
	b. LaMotte's minimum mean square procedures, 393	
	-i. Class C_0 : unrestricted, 393	
	-ii. Class C_1 : expectation of $y'Ay$ containing no β , 394	
	-iii. Class C_2 : translation-invariant, 394	
	-iv. Class C_3 : unbiased, 394	
	-v. Class C_4 : translation-invariant and unbiased, 394	
	c. Minimum variance estimation (MINVAR), 394	
	d. Minimum norm estimation (MINQUE), 397	
	e. REML, MINQUE and I-MINQUE, 398	
	f. REML for balanced data, 399	
	g. MINQUEO, 399	
	h. MINQUE for the 1-way classification, 399	
11.4	Summary	400
11.5	Exercises	402
12.	The Dispersion-Mean Model	405
12.1	The model	405
12.2	Ordinary least squares (OLS) yields MINQUEO	406
12.3	Fourth moments in the mixed model	407
	a. Dispersion matrix of $u \otimes u$, 407	
	-i. A normalizing transformation, 407	
	-ii. Example, 408	
	-iii. The general form of $E(ww' \otimes ww')$, 410	
	b. Fourth central moments of y , 411	
	-i. General case, 411	
	-ii. Under normality, 411	
	c. Dispersion matrix of \mathcal{Q} , 411	
	-i. General case, 411	
	-ii. Under normality, 412	
	d. Variance of a translation-invariant quadratic form, 412	
12.4	Generalized least squares (GLS)	413
	a. GLS yields REML equations under normality, 413	
	b. Excursus on estimating fixed effects, 415	
	c. REML is BLUE, 415	
12.5	Modified GLS yields ML	416
12.6	Balanced data	417
	a. Estimation under zero kurtosis, 417	
	-i. History, 417	
	-ii. The model, 419	
	-iii. Conclusion, 421	
	b. Estimation under non-zero kurtosis, 421	
	-i. The model, 421	
	-ii. ANOVA estimation, 422	
	-iii. Conclusion, 424	
12.7	Non-negative estimation	424
12.8	Summary	425
12.9	Exercises	426

Appendix F. Estimation Formulae for Unbalanced Data	427
PART I. THREE NESTED MODELS	427
F.1 The 1-way classification	427
a. Model, 427	
b. Analysis of variance estimators, 428	
c. Variances of analysis of variance estimators (under normality), 428	
d. Maximum likelihood estimation (under normality), 428	
e. Large-sample variances of maximum likelihood estimators (under normality), 428	
F.2 The 2-way nested classification	429
a. Model, 429	
b. Analysis of variance estimators, 429	
c. Variances of analysis of variance estimators (under normality), 429	
d. Large-sample variances of maximum likelihood estimators (under normality), 430	
F.3 The 3-way nested classification	431
a. Model, 431	
b. Analysis of variance estimators, 431	
c. Variances of analysis of variance estimators (under normality), 432	
PART II. THE 2-WAY CROSSED CLASSIFICATION	434
F.4 With interaction, random model	434
a. Model, 434	
b. Henderson Method I estimators, 434	
c. Variances of Henderson Method I estimators (under normality), 435	
d. Henderson Method III estimators, 437	
F.5 With interaction, mixed model	438
a. Model, 438	
b. Henderson Method III, 438	
F.6 No interaction, random model	439
a. Model, 439	
b. Henderson Method I, 439	
c. Variances of Henderson Method I estimators (under normality), 439	
d. Henderson Method III, 440	
e. Variances of Henderson Method III estimators (under normality), 440	
F.7 No interaction, mixed model	441
a. Model, 441	
b. Henderson Method III, 441	
Appendix M. Some Results in Matrix Algebra	442
M.1 Summing vectors, and J-matrices	442
M.2 Direct sums and products	443
M.3 A matrix notation in terms of elements	445
M.4 Generalized inverses	447
a. Definitions, 447	

	<i>b. Generalized inverses of $X'X$, 448</i>	
	<i>c. Partitioning $X'X$, 450</i>	
	<i>d. Rank results, 451</i>	
	<i>e. Vectors orthogonal to columns of X, 451</i>	
	<i>f. A theorem involving K' of maximum row rank for $K'X$ being null, 451</i>	
M.5	The Schur complement	453
M.6	The trace of a matrix	454
M.7	Differentiation of matrix expressions	454
	<i>a. Scalars, 454</i>	
	<i>b. Vectors, 455</i>	
	<i>c. Inner products, 455</i>	
	<i>d. Quadratic forms, 455</i>	
	<i>e. Inverses, 456</i>	
	<i>f. Determinants, 456</i>	
	<i>g. Traces, 457</i>	
M.8	The operators vec and vech	458
M.9	Vec permutation matrices	459
M.10	The equality $VV^{-1}X = X$	460
Appendix S. Some Results in Statistics		461
S.1	Conditional first and second moments	461
S.2	Least squares estimation	462
S.3	Normal and χ^2 -distributions	463
	<i>a. Central χ^2, 464</i>	
	<i>b. Mean squares, 464</i>	
	<i>c. Non-central χ^2, 465</i>	
S.4	<i>F</i> -distributions	465
S.5	Quadratic forms	466
S.6	Bayes estimation	467
	<i>a. Density functions, 467</i>	
	<i>b. Bayes Theorem, 468</i>	
	<i>c. Bayes estimation, 469</i>	
	<i>d. Example, 469</i>	
	<i>e. Empirical Bayes estimation, 471</i>	
S.7	Maximum likelihood	472
	<i>a. The likelihood function, 472</i>	
	<i>b. Maximum likelihood estimation, 472</i>	
	<i>c. Asymptotic dispersion matrix, 473</i>	
	<i>d. Transforming parameters, 474</i>	
References		475
List of Tables and Figures		490
Author Index		493
Subject Index		497