

Contents

CHAPTER 1. PROBABILITY

1

1. Borel's Normal Number Theorem, 1

*The Unit Interval—The Weak Law of Large Numbers—The Strong Law of Large Numbers—Strong Law Versus Weak—Length—The Measure Theory of Diophantine Approximation**

2. Probability Measures, 17

Spaces—Assigning Probabilities—Classes of Sets—Probability Measures—Lebesgue Measure on the Unit Interval—Sequence Space—Constructing σ -Fields**

3. Existence and Extension, 36

*Construction of the Extension—Uniqueness and the π - λ Theorem—Monotone Classes—Lebesgue Measure on the Unit Interval—Completeness—Nonmeasurable Sets—Two Impossibility Theorems**

4. Denumerable Probabilities, 51

General Formulas—Limit Sets—Independent Events—Subfields—The Borel–Cantelli Lemmas—The Zero–One Law

5. Simple Random Variables, 67

Definition—Convergence of Random Variables—Independence—Existence of Independent Sequences—Expected Value—Inequalities

6. The Law of Large Numbers, 85

The Strong Law—The Weak Law—Bernstein's Theorem—A Refinement of the Second Borel–Cantelli Lemma

*Stars indicate topics that may be omitted on a first reading.

7. Gambling Systems, 92

*Gambler's Ruin—Selection Systems—Gambling Policies—
Bold Play*—Timid Play**

8. Markov Chains, 111

*Definitions—Higher-Order Transitions—An Existence Theorem—
Transience and Persistence—Another Criterion for Persistence—
Stationary Distributions—Exponential Convergence*—Optimal
Stopping**

9. Large Deviations and the Law of the Iterated Logarithm,* 145

*Moment Generating Functions—Large Deviations—Chernoff's
Theorem—The Law of the Iterated Logarithm*

CHAPTER 2. MEASURE

158

10. General Measures, 158

*Classes of Sets—Conventions Involving ∞ —Measures—
Uniqueness*

11. Outer Measure, 165

Outer Measure—Extension—An Approximation Theorem

12. Measures in Euclidean Space, 171

*Lebesgue Measure—Regularity—Specifying Measures on the
Line—Specifying Measures in R^k —Strange Euclidean Sets**

13. Measurable Functions and Mappings, 182

*Measurable Mappings—Mappings into R^k —Limits
and Measureability—Transformations of Measures*

14. Distribution Functions, 187

*Distribution Functions—Exponential Distributions—Weak
Convergence—Convergence of Types*—Extremal Distributions**

CHAPTER 3. INTEGRATION

199

15. The Integral, 199

Definition—Nonnegative Functions—Uniqueness

16. Properties of the Integral, 206
Equalities and Inequalities—Integration to the Limit—Integration over Sets—Densities—Change of Variable—Uniform Integrability—Complex Functions
17. The Integral with Respect to Lebesgue Measure, 221
The Lebesgue Integral on the Line—The Riemann Integral—The Fundamental Theorem of Calculus—Change of Variable—The Lebesgue Integral in R^k —Stieltjes Integrals
18. Product Measure and Fubini's Theorem, 231
Product Spaces—Product Measure—Fubini's Theorem—Integration by Parts—Products of Higher Order
19. The L^p Spaces,* 241
Definitions—Completeness and Separability—Conjugate Spaces—Weak Compactness—Some Decision Theory—The Space L^2 —An Estimation Problem

CHAPTER 4. RANDOM VARIABLES AND EXPECTED VALUES

254

20. Random Variables and Distributions, 254
*Random Variables and Vectors—Subfields—Distributions—Multidimensional Distributions—Independence—Sequences of Random Variables—Convolution—Convergence in Probability—The Glivenko–Cantelli Theorem**
21. Expected Values, 273
Expected Value as Integral—Expected Values and Limits—Expected Values and Distributions—Moments—Inequalities—Joint Integrals—Independence and Expected Value—Moment Generating Functions
22. Sums of Independent Random Variables, 282
*The Strong Law of Large Numbers—The Weak Law and Moment Generating Functions—Kolmogorov's Zero–One Law—Maximal Inequalities—Convergence of Random Series—Random Taylor Series**

23. The Poisson Process, 297

Characterization of the Exponential Distribution—The Poisson Process—The Poisson Approximation—Other Characterizations of the Poisson Process—Stochastic Processes

24. The Ergodic Theorem,* 310

Measure-Preserving Transformations—Ergodicity—Ergodicity of Rotations—Proof of the Ergodic Theorem—The Continued-Fraction Transformation—Diophantine Approximation

CHAPTER 5. CONVERGENCE OF DISTRIBUTIONS

327

25. Weak Convergence, 327

Definitions—Uniform Distribution Modulo 1—Convergence in Distribution—Convergence in Probability—Fundamental Theorems—Helly's Theorem—Integration to the Limit*

26. Characteristic Functions, 342

*Definition—Moments and Derivatives—Independence—Inversion and the Uniqueness Theorem—The Continuity Theorem—Fourier Series**

27. The Central Limit Theorem, 357

*Identically Distributed Summands—The Lindeberg and Lyapounov Theorems—Dependent Variables**

28. Infinitely Divisible Distributions,* 371

Vague Convergence—The Possible Limits—Characterizing the Limit

29. Limit Theorems in R^k , 378

The Basic Theorems—Characteristic Functions—Normal Distributions in R^k —The Central Limit Theorem

30. The Method of Moments,* 388

The Moment Problem—Moment Generating Functions—Central Limit Theorem by Moments—Application to Sampling Theory—Application to Number Theory

- CHAPTER 6. DERIVATIVES AND CONDITIONAL PROBABILITY 400
31. Derivatives on the Line,* 400
The Fundamental Theorem of Calculus—Derivatives of Integrals—Singular Functions—Integrals of Derivatives—Functions of Bounded Variation
32. The Radon–Nikodym Theorem, 419
Additive Set Functions—The Hahn Decomposition—Absolute Continuity and Singularity—The Main Theorem
33. Conditional Probability, 427
The Discrete Case—The General Case—Properties of Conditional Probability—Difficulties and Curiosities—Conditional Probability Distributions
34. Conditional Expectation, 445
Definition—Properties of Conditional Expectation—Conditional Distributions and Expectations—Sufficient Subfields—Minimum-Variance Estimation**
35. Martingales, 458
*Definition—Submartingales—Gambling—Functions of Martingales—Stopping Times—Inequalities—Convergence Theorems—Applications: Derivatives—Likelihood Ratios—Reversed Martingales—Applications: de Finetti’s Theorem—Bayes Estimation—A Central Limit Theorem**
- CHAPTER 7. STOCHASTIC PROCESSES 482
36. Kolmogorov’s Existence Theorem, 482
Stochastic Processes—Finite-Dimensional Distributions—Product Spaces—Kolmogorov’s Existence Theorem—The Inadequacy of \mathcal{R}^T —A Return to Ergodic Theory—The Hewitt–Savage Theorem**
37. Brownian Motion, 498
Definition—Continuity of Paths—Measurable Processes—Irregularity of Brownian Motion Paths—The Strong Markov Property—The Reflection Principle—Skorohod Embedding—Invariance**

38. Nondenumerable Probabilities,* 526

*Introduction—Definitions—Existence Theorems—Consequences
of Separability*

APPENDIX	536
NOTES ON THE PROBLEMS	552
BIBLIOGRAPHY	581
LIST OF SYMBOLS	585
INDEX	587