

Contents

Preface	v
Reader's Guide	ix
List of Figures	xix
List of Tables	xxi
Prelude: Random Walks	1
The Model	1
Random variables	1
Probability	1
First calculations	3
Issues and Approaches	4
Issues	4
Approaches	4
Tools	5
Functionals of the Random Walk	6
Times of returns to the origin	7
Numbers of returns to the origin	9
First passage times	9
Maxima	11
Time spent positive	11
Limit Theorems	12
Summary	14
1 Probability	15
1.1 Random Experiments and Sample Spaces	16
1.1.1 Random experiments	16
1.1.2 Sample spaces	16
1.2 Events and Classes of Sets	17
1.2.1 Events	18

1.2.2	Basic set operations	18
1.2.3	Indicator functions	18
1.2.4	Operations on sequences of sets	20
1.2.5	Classes of sets closed under set operations	21
1.2.6	Generated classes	22
1.2.7	The monotone class theorem	23
1.2.8	Events, bis	23
1.3	Probabilities and Probability Spaces	23
1.3.1	Probability	23
1.3.2	Elementary properties	25
1.3.3	More advanced properties	26
1.3.4	Almost sure and null events	28
1.3.5	Uniqueness	28
1.4	Probabilities on \mathbb{R}	29
1.4.1	Distribution functions	29
1.4.2	Discrete probabilities	32
1.4.3	Absolutely continuous probabilities	33
1.4.4	Mixed distributions	34
1.5	Conditional Probability Given a Set	35
1.6	Complements	36
1.6.1	The extended real numbers	36
1.6.2	Measures	37
1.6.3	Lebesgue measure	38
1.6.4	Singular probabilities on \mathbb{R}	38
1.6.5	Representation of probabilities on \mathbb{R}	39
1.7	Exercises	40
2	Random Variables	43
2.1	Fundamentals	44
2.1.1	Random variables	44
2.1.2	Random vectors	45
2.1.3	Stochastic processes	45
2.1.4	Complex-valued random variables	45
2.1.5	The σ -algebra generated by a random variable	46
2.1.6	Simplified criteria	47
2.2	Combining Random Variables	47
2.2.1	Algebraic operations	47
2.2.2	Limiting operations	49
2.2.3	Transformations	50
2.2.4	Approximation of positive random variables	50
2.2.5	Monotone class theorems	51
2.3	Distributions and Distribution Functions	51
2.3.1	Random variables	52
2.3.2	Random vectors	53
2.4	Key Random Variables and Distributions	54

2.4.1	Discrete random variables	54
2.4.2	Absolutely continuous random variables	56
2.4.3	Random vectors	59
2.5	Transformation Theory	60
2.5.1	Random variables	60
2.5.2	Random vectors	62
2.6	Random Variables with Prescribed Distributions	63
2.6.1	Individual random variables	63
2.6.2	Random vectors	65
2.6.3	Sequences of random variables	65
2.7	Complements	65
2.7.1	Measurability with respect to sub- σ -algebras	65
2.7.2	Borel measurable functions	66
2.8	Exercises	67
3	Independence	71
3.1	Independent Random Variables	71
3.1.1	Fundamentals	71
3.1.2	Criteria for independence	71
3.1.3	Examples	75
3.2	Functions of Independent Random Variables	76
3.2.1	Transformation properties	76
3.2.2	Sums of independent random variables	77
3.3	Constructing Independent Random Variables	79
3.3.1	Finite families	79
3.3.2	Sequences	79
3.4	Independent Events	80
3.5	Occupancy Models	82
3.5.1	Four occupancy models	83
3.5.2	Occupancy numbers	84
3.5.3	Asymptotics	86
3.6	Bernoulli and Poisson Processes	88
3.6.1	Bernoulli processes	88
3.6.2	Poisson processes	91
3.7	Complements	94
3.7.1	Independent σ -algebras	94
3.7.2	Products of probability spaces	94
3.8	Exercises	95
4	Expectation	101
4.1	Definition and Fundamental Properties	102
4.1.1	Simple random variables	102
4.1.2	Positive random variables	103
4.1.3	Integrable random variables	107
4.1.4	Complex-valued random variables	109

4.2	Integrals with respect to Distribution Functions	110
4.2.1	Generalities	110
4.2.2	Discrete distribution functions	111
4.2.3	Absolutely continuous distribution functions	112
4.2.4	Mixed distribution functions	112
4.3	Computation of Expectations	113
4.3.1	Positive random variables	113
4.3.2	Integrable random variables	114
4.3.3	Functions of random variables	114
4.3.4	Functions of random vectors	115
4.3.5	Functions of independent random variables	117
4.3.6	Sums of independent random variables	118
4.4	L^p Spaces and Inequalities	119
4.4.1	L^p spaces	119
4.4.2	Key inequalities	119
4.5	Moments	123
4.5.1	Moments of random variables	123
4.5.2	Variance and standard deviation	124
4.5.3	Covariance and correlation	125
4.5.4	Moments of random vectors	126
4.5.5	Multivariate normal distributions	126
4.6	Complements	127
4.6.1	Integration with respect to Lebesgue measure	127
4.6.2	Expectation for product probabilities	128
4.7	Exercises	130
5	Convergence of Sequences of Random Variables	135
5.1	Modes of Convergence	135
5.1.1	Convergence of random variables as functions	135
5.1.2	Convergence of distribution functions	136
5.1.3	Alternative criteria	137
5.2	Relationships Among the Modes	140
5.2.1	Implications always valid	140
5.2.2	Counterexamples	141
5.2.3	Implications of restricted validity	142
5.2.4	Implications involving subsequences	144
5.3	Convergence under Transformations	145
5.3.1	Algebraic operations	145
5.3.2	Continuous mappings	148
5.4	Convergence of Random Vectors	149
5.4.1	Convergence of random vectors as functions	149
5.4.2	Convergence in distribution	149
5.4.3	Continuous mappings	150
5.5	Limit Theorems for Bernoulli Summands	150
5.5.1	Laws of large numbers	151

5.5.2	Central limit theorems	152
5.5.3	The Poisson limit theorem	155
5.5.4	Approximation of continuous functions	156
5.6	Complements	157
5.6.1	L^p Convergence of random variables	157
5.7	Exercises	158
6	Characteristic Functions	163
6.1	Definition and Basic Properties	163
6.1.1	Fundamentals	163
6.1.2	Elementary properties	164
6.2	Inversion and Uniqueness Theorems	166
6.2.1	The inversion theorem	166
6.2.2	The uniqueness theorem	167
6.2.3	Specialized inversion theorems	167
6.3	Moments and Taylor Expansions	169
6.3.1	Calculation of moments known to exist	169
6.3.2	Establishing existence of moments	170
6.3.3	Taylor expansions of characteristic functions	171
6.4	Continuity Theorems and Applications	171
6.4.1	Convergence in distribution	171
6.4.2	The Lévy continuity theorem	172
6.4.3	Application to classical limit theorems	173
6.5	Other Transforms	174
6.5.1	Characteristic functions of random vectors	175
6.5.2	Laplace transforms	176
6.5.3	Moment generating functions	177
6.5.4	Generating functions	177
6.6	Complements	178
6.6.1	Helly's theorem	178
6.7	Exercises	179
7	Classical Limit Theorems	183
7.1	Series of Independent Random Variables	183
7.1.1	Kolmogorov's inequality	183
7.1.2	The three series theorem	185
7.2	The Strong Law of Large Numbers	187
7.3	The Central Limit Theorem	190
7.3.1	The Lyapunov condition	191
7.3.2	The Lindeberg condition	192
7.4	The Law of the Iterated Logarithm	196
7.4.1	Normally distributed summands	197
7.4.2	More general versions	200
7.5	Applications of the Limit Theorems	200
7.5.1	Monte Carlo integration	201

7.5.2	Maximum likelihood estimation	201
7.5.3	Empirical distribution functions	205
7.5.4	Random sums of independent random variables	207
7.5.5	Renewal processes	208
7.6	Complements	210
7.6.1	The Berry-Essén theorem	210
7.7	Exercises	212
8	Prediction and Conditional Expectation	217
8.1	Prediction in L^2	218
8.1.1	The inner product and norm	218
8.1.2	L^2 as metric space	219
8.1.3	Orthogonality and orthonormality	221
8.1.4	The orthogonal decomposition theorem	222
8.1.5	Computation of MMSE predictors	224
8.1.6	Linear prediction	224
8.2	Conditional Expectation Given a Finite Set of Random Variables	225
8.2.1	Basics	225
8.2.2	Examples	226
8.2.3	Conditional probability	227
8.3	Conditional Expectation for $X \in L^2$	227
8.3.1	Conditional expectation as MMSE prediction	227
8.3.2	Properties of conditional expectation	228
8.4	Positive and Integrable Random Variables	230
8.5	Conditional Distributions	233
8.5.1	Generalities	233
8.5.2	Discrete random variables	234
8.5.3	Absolutely continuous random variables	235
8.6	Computational Techniques	236
8.6.1	General results	236
8.6.2	Special cases	238
8.7	Complements	238
8.7.1	Mixed conditional distributions	238
8.7.2	Conditional expectation given a σ -algebra	239
8.8	Exercises	239
9	Martingales	243
9.1	Fundamentals	243
9.1.1	Definitions	243
9.1.2	Examples	245
9.1.3	Compositions and transformations	247
9.2	Stopping Times	248
9.3	Optional Sampling Theorems	250
9.3.1	Optional sampling theorems for martingales	251

9.3.2	Applications of optional sampling theorems	253
9.4	Martingale Convergence Theorems	255
9.4.1	Upcrossings and almost sure convergence	255
9.4.2	Almost sure convergence of submartingales	256
9.4.3	Almost sure convergence of martingales	258
9.4.4	Uniformly integrable martingales	259
9.5	Applications of Convergence Theorems	260
9.5.1	The Radon-Nikodym theorem	260
9.5.2	Zero-one laws	262
9.5.3	Likelihood ratios	264
9.6	Complements	264
9.6.1	Conditioning on Y_0, \dots, Y_T	264
9.6.2	Martingales with respect to filtrations	267
9.6.3	Reversed martingales	267
9.7	Exercises	268
A	Notation	271
B	Named Objects	275
	Bibliography	277
	Index	279