

Contents

Preface	vii
0 Basic Concepts	1
0.1 Weak Formulation of Boundary Value Problems	1
0.2 Ritz-Galerkin Approximation	3
0.3 Error Estimates	4
0.4 Piecewise Polynomial Spaces – The Finite Element Method	7
0.5 Relationship to Difference Methods	9
0.6 Computer Implementation of Finite Element Methods	10
0.7 Local Estimates	12
0.8 Weighted Norm Estimates	13
0.x Exercises	18
1 Sobolev Spaces	21
1.1 Review of Lebesgue Integration Theory	21
1.2 Generalized (weak) Derivatives	24
1.3 Sobolev Norms and Associated Spaces	27
1.4 Inclusion Relations and Sobolev’s Inequality	30
1.5 Review of Chapter 0	33
1.6 Trace Theorems	34
1.7 Negative Norms and Duality	38
1.x Exercises	40
2 Variational Formulation of Elliptic Boundary Value Problems	47
2.1 Inner-Product Spaces	47
2.2 Hilbert Spaces	49
2.3 Projections onto Subspaces	50

2.4 Riesz Representation Theorem	53
2.5 Formulation of Symmetric Variational Problems	54
2.6 Formulation of Nonsymmetric Variational Problems	57
2.7 The Lax-Milgram Theorem	58
2.8 Estimates for General Finite Element Approximation	62
2.9 Higher-dimensional Examples	63
2.x Exercises	65
3 The Construction of a Finite Element Space	67
3.1 The Finite Element	67
3.2 Triangular Finite Elements	69
The Lagrange Element	70
The Hermite Element	73
The Argyris Element	74
3.3 The Interpolant	75
3.4 Equivalence of Elements	79
3.5 Rectangular Elements	82
Tensor Product Elements	83
The Serendipity Element	84
3.6 Higher-dimensional Elements	85
3.7 Exotic Elements	87
3.x Exercises	87
4 Polynomial Approximation Theory in Sobolev Spaces	91
4.1 Averaged Taylor Polynomials	91
4.2 Error Representation	94
4.3 Bounds for Riesz Potentials	98
4.4 Bounds for the Interpolation Error	103
4.5 Inverse Estimates	109
4.6 Tensor-product Polynomial Approximation	112
4.7 Isoparametric Polynomial Approximation	117
4.8 Interpolation of Non-smooth Functions	118
4.x Exercises	120
5 n-Dimensional Variational Problems	123
5.1 Variational Formulation of Poisson's Equation	123
5.2 Variational Formulation of the Pure Neumann Problem	126
5.3 Coercivity of the Variational Problem	128

5.4 Variational Approximation of Poisson’s Equation	130
5.5 Elliptic Regularity Estimates	133
5.6 General Second-Order Elliptic Operators	135
5.7 Variational Approximation of General Elliptic Problems	138
5.8 Negative-Norm Estimates	140
5.9 The Plate-Bending Biharmonic Problem	142
5.x Exercises	146
6 Finite Element Multigrid Methods	149
6.1 A Model Problem	149
6.2 Mesh-Dependent Norms	150
6.3 The Multigrid Algorithm	152
6.4 Approximation Property	155
6.5 \mathcal{W} -cycle Convergence for the k^{th} Level Iteration	156
6.6 \mathcal{V} -cycle Convergence for the k^{th} Level Iteration	159
6.7 Full Multigrid Convergence Analysis and Work Estimates	164
6.x Exercises	166
7 Max-norm Estimates	169
7.1 Main Theorem	169
7.2 Reduction to Weighted Estimates	172
7.3 Proof of Lemma 7.2.6	173
7.4 Proof of Lemmas 7.3.7 and 7.3.11	178
7.5 L^p Estimates (Regular Coefficients)	182
7.6 L^p Estimates (Irregular Coefficients)	184
7.7 A Nonlinear Example	188
7.x Exercises	191
8 Variational Crimes	195
8.1 Departure from the Framework	196
8.2 Finite Elements with Interpolated Boundary Conditions	198
8.3 Nonconforming Finite Elements	205
8.4 Isoparametric Finite Elements	208
8.x Exercises	211
9 Applications to Planar Elasticity	217
9.1 The Boundary Value Problems	217
9.2 Weak Formulation and Korn’s Inequality	219

9.3 Finite Element Approximation and Locking	226
9.4 A Robust Method for the Pure Displacement Problem	229
9.x Exercises	233
10 Mixed Methods	237
10.1 Examples of Mixed Variational Formulations	237
10.2 Abstract Mixed Formulation	239
10.3 Discrete Mixed Formulation	242
10.4 Convergence Results for Velocity Approximation	244
10.5 The Discrete Inf-Sup Condition	247
10.6 Verification of the Inf-Sup Condition	253
10.x Exercises	259
11 Iterative Techniques for Mixed Methods	261
11.1 Iterated Penalty Method	261
11.2 Stopping Criteria	265
11.3 Augmented Lagrangian Method	267
11.4 Application to the Navier-Stokes Equations	269
11.5 Computational Examples	272
11.x Exercises	275
12 Applications of Operator-Interpolation Theory	277
12.1 The Real Method of Interpolation	277
12.2 Real Interpolation of Sobolev Spaces	279
12.3 Finite Element Convergence Estimates	282
12.4 The Simultaneous Approximation Theorem	284
12.5 Precise Characterizations of Regularity	285
12.x Exercises	286
References	287
Index	291