

CONTENTS

CHAPTER 1 An introduction to the theory of Volterra equations	1
1.1. Preliminaries	1
1.1.1. Classification of Volterra equations	1
1.1.2. Connections with ordinary differential equations	2
1.1.3. Connections with partial differential equations	5
1.2. A brief history of Volterra integral equations	7
1.3. Existence and uniqueness of solutions	11
1.3.1. Linear Volterra integral equations of the second kind	11
1.3.2. Linear Volterra integro-differential equations	18
1.3.3. Linear Volterra integral equations of the first kind	20
1.3.4. Nonlinear Volterra equations	22
1.3.5. Smoothness of solutions	28
1.4. Some results on the asymptotic behavior of Volterra equations	35
1.5. Gronwall-type inequalities and comparison theorems	38
1.5.1. Integral inequalities of Gronwall type	38
1.5.2. A comparison theorem	40
1.5.3. Discrete Gronwall-type inequalities	40
Notes	45
CHAPTER 2 Numerical quadrature	51
2.1. General theory	51
2.1.1. The approximation error	51
2.1.2. Interpolatory quadrature formulas	54
2.2. Newton-Cotes formulas	56
2.3. Gauss-Legendre formulas	59
2.4. Radau and Lobatto formulas	60
2.4.1. Radau formulas	60
2.4.2. Lobatto formulas	62
2.5. Summary of error terms	63

2.6.	(ρ, σ)-reducible quadrature formulas	64
2.6.1.	Linear multistep methods	67
2.6.2.	Construction of quadrature weights	68
2.6.3.	The Gregory quadrature rules	73
2.6.4.	Linear difference equations	76
2.6.5.	Asymptotic behavior of the weights	79
2.6.6.	The approximation error	80
2.6.7.	Quadrature based on backward differentiation formulas	82
2.7.	Repeated quadrature formulas	84
2.8.	Product integration	86
Notes		91
CHAPTER 3 Linear methods for Volterra equations		93
3.1.	Brief survey of early discretization methods	93
3.2.	Volterra integral equations of the second kind	96
3.2.1.	Direct quadrature methods	96
3.2.2.	Numerical examples	99
3.2.3.	(ρ, σ)-reducible methods	100
3.2.4.	A class of linear methods: VLM methods	101
3.2.5.	Consistency of VLM methods	105
3.2.6.	Special VLM formulas	108
3.2.7.	Numerical examples	113
3.2.8.	Convergence results for VLM methods	115
3.2.9.	Numerical examples	123
3.2.10.	Extrapolation methods	128
3.2.11.	Numerical examples	131
3.2.12.	Block-by-block methods	133
3.3.	Volterra integral equations of the first kind	135
3.3.1.	VLM methods	135
3.3.2.	Consistency of VLM methods	137
3.3.3.	Special VLM formulas	139
3.3.4.	Convergence results for VLM methods	141
3.3.5.	Local differentiation methods	147
3.3.6.	Numerical examples	148
3.4.	Volterra integro-differential equations	151
3.4.1.	VLM methods	151
3.4.2.	Consistency of VLM methods	153
3.4.3.	Convergence results for VLM methods	154

3.4.4. Numerical examples	157
3.5. Starting formulas	159
3.5.1. Explicit starting formulas	160
3.5.2. Implicit starting formulas	162
Notes	163
CHAPTER 4 Runge-Kutta-type methods for Volterra equations	169
4.1. Runge-Kutta methods for integral equations of the second kind	169
4.1.1. Introduction	169
4.1.2. Historical notes	177
4.1.3. On the order conditions for RK methods in ODEs	178
4.1.4. Order conditions for VRK formulas	180
4.1.5. PVRK formulas	186
4.1.6. BVRK formulas	189
4.1.7. Lag term formulas	196
4.1.8. Convergence results	202
4.1.9. Summary	209
4.1.10. Numerical examples	210
4.2. Runge-Kutta methods for integro-differential equations	213
4.2.1. Introduction	213
4.2.2. Order conditions for VDRK formulas	219
4.2.3. PVDRK formulas	221
4.2.4. BVDRK formulas	223
4.2.5. Convergence results	227
4.2.6. Numerical examples	229
Notes	231
CHAPTER 5 Collocation methods for Volterra equations with regular kernels	235
5.1. The polynomial spline spaces $S_m^{(d)}(Z_N)$	235
5.2. Collocation methods for ordinary differential equations	240
5.2.1. Connection with Runge-Kutta methods	240
5.2.2. Connection with Runge-Kutta-Nyström methods	241
5.2.3. Approximations in $S_m^{(-1)}(Z_N)$	242
5.3. Polynomial spline collocation for second-kind integral equations	247
5.3.1. The collocation equation	247
5.3.2. Global convergence results	251

5.3.3. Local superconvergence on \bar{Z}_N	254
5.3.4. Convergence results for the discretized collocation equation	259
5.3.5. Examples	265
5.3.6. Iterated collocation	269
5.3.7. Numerical examples	277
5.4. Polynomial spline collocation for integro-differential equations	279
5.4.1. Global convergence results	279
5.4.2. Local superconvergence on \bar{Z}_N	284
5.4.3. Convergence results for the fully discretized collocation equation	288
5.4.4. Examples of optimal methods	295
5.4.5. Numerical examples	297
5.5. Polynomial spline collocation for integral equations of the first kind	298
5.5.1. Global convergence results	298
5.5.2. Local differentiation formulas based on collocation	304
5.5.3. Convergence results for the fully discretized collocation equation	308
5.5.4. Examples	312
5.5.5. Numerical examples	314
Notes	317
CHAPTER 6 Volterra equations with weakly singular kernels	323
6.1. Fractional linear multistep methods	323
6.1.1. Introduction	323
6.1.2. Convolution quadratures for integrals of fractional order	325
6.1.3. Computation of the fractional convolution quadrature weights	335
6.1.4. Fractional linear multistep methods for second-kind equations	342
6.2. Polynomial spline collocation for second-kind equations	347
6.2.1. The approximation power of polynomial spline functions	348
6.2.2. Polynomial spline collocation on quasi-uniform meshes	352
6.2.3. Polynomial spline collocation on graded meshes	360
6.2.4. Discretization of the collocation equation	364
6.2.5. Collocation in nonpolynomial spline spaces	371
6.2.6. Product integration and iterated collocation methods	375
6.3. Polynomial spline collocation for integro-differential equations	379
6.3.1. Polynomial spline collocation on quasi-uniform meshes	379
6.3.2. Polynomial spline collocation on graded meshes	386
6.3.3. Discretization of the collocation equation	388

6.4.	First-kind integral equations with weakly singular kernels	391
6.4.1.	Polynomial spline collocation and product integration methods	391
6.4.2.	Convergence results and open problems	396
6.4.3.	First-kind Abel-type integral equations with nonexact data	398
Notes		403
CHAPTER 7 Numerical stability		409
7.1.	Stability of the exact solution	409
7.1.1.	Definition of stability	410
7.1.2.	Differential equations with perturbed right-hand side	412
7.1.3.	The logarithmic norm	415
7.1.4.	Polynomial convolution kernels	417
7.1.5.	Finitely decomposable kernels	422
7.2.	Stability of numerical methods	428
7.2.1.	Weak and strong stability	428
7.2.2.	Block-stability	430
7.2.3.	Local stability	433
7.3.	VLM methods for integral equations	434
7.3.1.	Preliminaries	434
7.3.2.	Stability results for the basic test equation	436
7.3.3.	Stability results for the convolution test equation	442
7.3.4.	Stability results for polynomial convolution kernels	453
7.3.5.	Stability results for finitely decomposable kernels	455
7.3.6.	Stability results for positive definite convolution kernels	463
7.3.7.	Numerical examples	464
7.4.	VLM methods for integro-differential equations	469
7.4.1.	Stability results for the basic test equation	469
7.4.2.	Numerical examples	471
7.5.	VRK methods for second-kind integral equations	472
7.5.1.	Recurrence relations for the convolution test equation	473
7.5.2.	Stability results for the basic test equation	476
7.5.3.	Stability results for the convolution test equation	481
7.5.4.	Stability results for positive definite convolution kernels	487
7.5.5.	Numerical examples	488
7.6.	VDRK methods for integro-differential equations	489
7.6.1.	Recurrence relations for the basic test equation	489
7.6.2.	Extended methods	490
7.6.3.	Mixed methods	491

7.6.4. Application of VRK methods	491
7.6.5. Numerical examples	492
Notes	493
 CHAPTER 8 Software and test examples 499	
8.1. Introduction	499
8.2. Routines for second-kind Volterra integral equations	499
8.2.1. Summary of published routines	501
8.2.2. Test problems	505
8.3. Routines for Volterra integro-differential equations	512
8.3.1. Summary of published routines	512
8.3.2. Test problems	513
8.4. Routines for first-kind Volterra integral equations	514
8.4.1. Summary of published routines	515
8.4.2. Test problems	517
Notes	519
References	521
Notations	573
Index	577