

Volume II

Table of Contents

Volume I: Table of Contents	xiii
Chapter 5	
The Mathematical Theory of Iterative Methods	1
5.1 Several results from functional analysis	1
5.1.1 Linear spaces	1
5.1.2 Operators in linear normed spaces	5
5.1.3 Operators in a Hilbert space	8
5.1.4 Functions of a bounded operator	14
5.1.5 Operators in a finite-dimensional space.....	15
5.1.6 The solubility of operator equations	18
5.2 Difference schemes as operator equations	21
5.2.1 Examples of grid-function spaces	21
5.2.2 Several difference identities	25
5.2.3 Bounds for the simplest difference operators	28
5.2.4 Lower bounds for certain difference operators	31
5.2.5 Upper bounds for difference operators.....	41
5.2.6 Difference schemes as operator equations in abstract spaces	42
5.2.7 Difference schemes for elliptic equations with constant coefficients	47
5.2.8 Equations with variable coefficients and with mixed derivatives	50
5.3 Basic concepts from the theory of iterative methods	55
5.3.1 The steady state method	55
5.3.2 Iterative schemes	56
5.3.3 Convergence and iteration counts.....	58
5.3.4 Classification of iterative methods	60

Chapter 6	
Two-Level Iterative Methods	65
6.1 Choosing the iterative parameters	65
6.1.1 The initial family of iterative schemes	65
6.1.2 The problem for the error	66
6.1.3 The self-adjoint case	67
6.2 The Chebyshev two-level method	69
6.2.1 Construction of the set of iterative parameters	69
6.2.2 On the optimality of the <i>a priori</i> estimate	71
6.2.3 Sample choices for the operator D	72
6.2.4 On the computational stability of the method	75
6.2.5 Construction of the optimal sequence of iterative parameters	82
6.3 The simple iteration method	86
6.3.1 The choice of the iterative parameter	86
6.3.2 An estimate for the norm of the transformation operator ..	88
6.4 The non-self-adjoint case. The simple iteration method	90
6.4.1 Statement of the problem	90
6.4.2 Minimizing the norm of the transformation operator ..	91
6.4.3 Minimizing the norm of the resolving operator	98
6.4.4 The symmetrization method	104
6.5 Sample applications of the iterative methods	105
6.5.1 A Dirichlet difference problem for Poisson's equation in a rectangle	105
6.5.2 A Dirichlet difference problem for Poisson's equation in an arbitrary region	110
6.5.3 A Dirichlet difference problem for an elliptic equation with variable coefficients	116
6.5.4 A Dirichlet difference problem for an elliptic equation with mixed derivatives	122
Chapter 7	
Three-Level Iterative Methods	125
7.1 An estimate of the convergence rate	125
7.1.1 The basic family of iterative schemes	125
7.1.2 An estimate for the norm of the error	126
7.2 The Chebyshev semi-iterative method	129
7.2.1 Formulas for the iterative parameters	129
7.2.2 Sample choices for the operator D	132
7.2.3 The algorithm of the method	132

7.3 The stationary three-level method	133
7.3.1 The choice of the iterative parameters	133
7.3.2 An estimate for the rate of convergence	134
7.4 The stability of two-level and three-level methods relative to <i>a priori</i> data.....	136
7.4.1 Statement of the problem	136
7.4.2 Estimates for the convergence rates of the methods	138
 Chapter 8	
Iterative Methods of Variational Type	145
8.1 Two-level gradient methods	145
8.1.1 The choice of the iterative parameters	145
8.1.2 A formula for the iterative parameters	147
8.1.3 An estimate of the convergence rate	149
8.1.4 Optimality of the estimate in the self-adjoint case	151
8.1.5 An asymptotic property of the gradient methods in the self-adjoint case	153
8.2 Examples of two-level gradient methods	156
8.2.1 The steepest-descent method	156
8.2.2 The minimal residual method	158
8.2.3 The minimal correction method	160
8.2.4 The minimal error method	161
8.2.5 A sample application of two-level methods	162
8.3 Three-level conjugate-direction methods	164
8.3.1 The choice of the iterative parameters. An estimate of the convergence rate	164
8.3.2 Formulas for the iterative parameters. The three-level iterative scheme	167
8.3.3 Variants of the computational formulas	172
8.4 Examples of the three-level methods	174
8.4.1 Special cases of the conjugate-direction methods	174
8.4.2 Locally optimal three-level methods	176
8.5 Accelerating the convergence of two-level methods in the self-adjoint case	181
8.5.1 An algorithm for the acceleration process	181
8.5.2 An estimate of the effectiveness	183
8.5.3 An example	184

Chapter 9

Triangular Iterative Methods	189
9.1 The Gauss-Seidel method	189
9.1.1 The iterative scheme for the method	189
9.1.2 Sample applications of the method	193
9.1.3 Sufficient conditions for convergence	196
9.2 The successive over-relaxation method	199
9.2.1 The iterative scheme. Sufficient conditions for convergence	199
9.2.2 The choice of the iterative parameter	200
9.2.3 An estimate of the spectral radius	204
9.2.4 A Dirichlet difference problem for Poisson's equation in a rectangle	207
9.2.5 A Dirichlet difference problem for an elliptic equation with variable coefficients	212
9.3 Triangular methods	215
9.3.1 The iterative scheme	215
9.3.2 An estimate of the convergence rate	217
9.3.3 The choice of the iterative parameter	219
9.3.4 An estimate for the convergence rates of the Gauss-Seidel and relaxation methods	220

Chapter 10

The Alternate-Triangular Method	225
10.1 The general theory of the method	225
10.1.1 The iterative scheme	225
10.1.2 Choice of the iterative parameters	228
10.1.3 A method for finding δ and Δ	231
10.1.4 A Dirichlet difference problem for Poisson's equation in a rectangle	233
10.2 Boundary-value difference problems for elliptic equations in a rectangle	241
10.2.1 A Dirichlet problem for an equation with variable coefficients	241
10.2.2 A modified alternate-triangular method	243
10.2.3 A comparison of the variants of the method	251
10.2.4 A boundary-value problem of the third kind	252
10.2.5 A Dirichlet difference problem for an equation with mixed derivatives	255

10.3 The alternate-triangular method for elliptic equations in arbitrary regions	258
10.3.1 The statement of the difference problem	258
10.3.2 The construction of an alternate-triangular method	259
10.3.3 A Dirichlet problem for Poisson's equation in an arbitrary region	264
 Chapter 11	
The Alternating-Directions Method	269
11.1 The alternating-directions method in the commutative case	269
11.1.1 The iterative scheme for the method	269
11.1.2 The choice of the parameters	271
11.1.3 A fractionally-linear transformation	273
11.1.4 The optimal set of parameters	276
11.2 Sample applications of the method	280
11.2.1 A Dirichlet difference problem for Poisson's equation in a rectangle	280
11.2.2 A boundary-value problem of the third kind for an elliptic equation with separable variables	285
11.2.3 A high-accuracy Dirichlet difference problem	289
11.3 The alternating-directions method in the general case	294
11.3.1 The case of non-commuting operators	294
11.3.2 A Dirichlet difference problem for an elliptic equation with variable coefficients	297
 Chapter 12	
Methods for Solving Equations with Indefinite and Singular Operators	303
12.1 Equations with real indefinite operators	303
12.1.1 The iterative scheme. The choice of the iterative parameters	303
12.1.2 Transforming the operator in the self-adjoint case	306
12.1.3 The iterative method with the Chebyshev parameters	309
12.1.4 Iterative methods of variational type	313
12.1.5 Examples	315
12.2 Equations with complex operators	317
12.2.1 The simple iteration method	317
12.2.2 The alternating-directions method	321
12.3 General iterative methods for equations with singular operators	326
12.3.1 Iterative schemes in the case of a non-singular operator B .	326
12.3.2 The minimum-residual iterative method	330
12.3.3 A method with the Chebyshev parameters	333

12.4 Special methods	339
12.4.1 A Neumann difference problem for Poisson's equation in a rectangle	339
12.4.2 A direct method for the Neumann problem	343
12.4.3 Iterative schemes with a singular operator B	346
 Chapter 13	
Iterative Methods for Solving Non-Linear Equations	351
13.1 Iterative methods. The general theory	351
13.1.1 The simple iteration method for equations with a monotone operator	351
13.1.2 Iterative methods for the case of a differentiable operator	354
13.1.3 The Newton-Kantorovich method	358
13.1.4 Two-stage iterative methods	362
13.1.5 Other iterative methods	365
13.2 Methods for solving non-linear difference schemes	368
13.2.1 A difference scheme for a one-dimensional elliptic quasi-linear equation	368
13.2.2 The simple iteration method	377
13.2.3 Iterative methods for quasi-linear elliptic difference equations in a rectangle	379
13.2.4 Iterative methods for weakly-nonlinear equations	385
 Chapter 14	
Example Solutions of Elliptic Grid Equations	389
14.1 Methods for constructing implicit iterative schemes	389
14.1.1 The regularizer principle in the general theory of iterative methods	389
14.1.2 Iterative schemes with a factored operator	393
14.1.3 A method for implicitly inverting the operator B (a two-stage method)	399
14.2 Examples of solving elliptic boundary-value problems	401
14.2.1 Direct and iterative methods	401
14.2.2 A high-accuracy Dirichlet difference problem in the multi-dimensional case	407
14.2.3 A boundary-value problem of the third kind for an equation with mixed derivatives in a rectangle	409
14.2.4 Iterative methods for solving a difference problem	414
14.3 Systems of elliptic equations	423
14.3.1 A Dirichlet problem for systems of elliptic equations in a p -dimensional parallelepiped	423
14.3.2 A system of equations from elasticity theory	428

14.4 Methods for solving elliptic equations in irregular regions	431
14.4.1 Difference problems in regions of complex form, and methods for their solution	431
14.4.2 Decomposition of regions	433
14.4.3 An algorithm for the domain decomposition method	438
14.4.4 The method of domain augmentation to a rectangle	442
Chapter 15	
Methods for Solving Elliptic Equations in Curvilinear Orthogonal Coordinates	447
15.1 Posing boundary-value problems for differential equations	447
15.1.1 Elliptic equations in a cylindrical system of coordinates	447
15.1.2 Boundary-value problems for equations in a cylindrical coordinate system	450
15.2 The solution of difference problems in cylindrical coordinates	454
15.2.1 Difference schemes without mixed derivatives in the axially-symmetric case	454
15.2.2 Direct methods	459
15.2.3 The alternating-directions method	461
15.2.4 The solution of equations defined on the surface of a cylinder	465
15.3 Solution of difference problems in polar coordinate systems	470
15.3.1 Difference schemes for equations in a circle or a ring	470
15.3.2 The solubility of the boundary-value difference problems	473
15.3.3 The superposition principle for a problem in a circle	476
15.3.4 Direct methods for solving equations in a circle or a ring	478
15.3.5 The alternating-directions method	479
15.3.6 Solution of difference problems in a ring sector	483
15.3.7 The general variable-coefficients case	485
Appendices	
Construction of the minimax polynomial	489
Bibliography	495
Translator's note	497
Index	499