

Contents

Chapter I

Mathematical preliminaries and some general principles

	page
Some notes on the numerical examples	1
§ 1. Introduction to problems involving differential equations	2
1.1. Initial-value and boundary-value problems in ordinary differential equations	2
1.2. Linear boundary-value problems	3
1.3. Problems in partial differential equations	5
§ 2. Finite differences and interpolation formulae	6
2.1. Difference operators and interpolation formulae	6
2.2. Some integration formulae which will be needed later	9
2.3. Repeated integration	11
2.4. Calculation of higher derivatives	17
2.5. HERMITE's generalization of TAYLOR's formula.	19
§ 3. Further useful formulae from analysis	21
3.1. GAUSS's and GREEN's formulae for two independent variables	21
3.2. Corresponding formulae for more than two independent variables .	22
3.3. Co-normals and boundary-value problems in elliptic differential equations	23
3.4. GREEN's functions	25
3.5. Auxiliary formulae for the biharmonic operator	26
§ 4. Some error distribution principles	28
4.1. General approximation. "Boundary" and "interior" methods	28
4.2. Collocation, least-squares, orthogonality method, partition method, relaxation	29
4.3. The special case of linear boundary conditions	31
4.4. Combination of iteration and error distribution	33
§ 5. Some useful results from functional analysis	34
5.1. Some basic concepts of functional analysis with examples	34
5.2. The general theorem on iterative processes	36
5.3. The operator T applied to boundary-value problems	38
5.4. Problems of monotonic type	42
5.5. Application to systems of linear equations of monotonic type	43
5.6. Non-linear boundary-value problems	47

Chapter II

Initial-value problems in ordinary differential equations

§ 1. Introduction	48
1.1. The necessity for numerical methods	48
1.2. Accuracy in the numerical solution of initial-value problems	49

	page
1.3. Some general observations on error estimation for initial-value problems	50
I. Comparison of two approximations with different lengths of step	51
II. The terminal check	52
1.4. Differential equations of the first order. Preliminaries	52
1.5. Some methods of integration	53
1.6. Error estimation	57
I. Polygon method	58
1.7. Corresponding error estimates for the improved methods	59
II. Improved polygon method	59
III. Improved Euler-Cauchy method	60
§ 2. The Runge-Kutta method for differential equations of the n -th order	61
2.1. A general formulation	62
2.2. The special Runge-Kutta formulation	64
2.3. Derivation of the Runge-Kutta formulae	66
2.4. Hints for using the Runge-Kutta method	68
2.5. Terminal checks and iteration methods	72
2.6. Examples	73
§ 3. Finite-difference methods for differential equations of the first order	78
3.1. Introduction	78
3.2. Calculation of starting values	79
3.3. Formulae for the main calculation	82
I. The Adams extrapolation method	83
II. The Adams interpolation method	85
III. Central-difference interpolation method	86
IV. Mixed extrapolation and interpolation methods	88
3.4. Hints for the practical application of the finite-difference methods	88
3.5. Examples	91
3.6. Differential equations in the complex plane	95
3.7. Implicit differential equations of the first order	96
§ 4. Theory of the finite-difference methods	97
4.1. Convergence of the iterations in the main calculation	97
4.2. Convergence of the starting iteration	99
4.3. Recursive error estimates	101
4.4. Independent error estimates	105
4.5. Error estimates for the starting iteration	109
4.6. Systems of differential equations	110
4.7. Instability in finite-difference methods	111
4.8. Improvement of error estimates by use of a weaker Lipschitz condition	113
4.9. Error estimation by means of the general theorem on iteration	114
§ 5. Finite-difference methods for differential equations of higher order	116
5.1. Introduction	116
5.2. Calculation of starting values	118
5.3. Iterative calculation of starting values for the second-order equation $y'' = f(x, y, y')$	118
5.4. Extrapolation methods	123
5.5. Interpolation methods	126
5.6. Convergence of the iteration in the main calculation	131
5.7. Principle of an error estimate for the main calculation	133
5.8. Instability of finite-difference methods	135

	page
5.9. Reduction of initial-value problems to boundary-value problems	136
5.10. Miscellaneous exercises on Chapter II	137
5.11. Solutions	138

Chapter III

Boundary-value problems in ordinary differential equations

§ 1. The ordinary finite-difference method	141
1.1. Description of the finite-difference method	141
1.2. Examples of boundary-value problems of the second order	143
I. A linear boundary-value problem of the second order	143
II. A non-linear boundary-value problem of the second order	145
III. An eigenvalue problem	147
IV. Infinite interval	150
1.3. A linear boundary-value problem of the fourth order	152
1.4. Relaxation	154
I. A linear boundary-value problem	155
II. A non-linear boundary-value problem	159
§ 2. Refinements of the ordinary finite-difference method	160
2.1. Improvement by using finite expressions which involve more pivotal values	160
2.2. Derivation of finite expressions	161
2.3. The finite-difference method of a higher approximation	163
2.4. Basic formulae for Hermitian methods	164
2.5. The Hermitian method in the general case	166
2.6. Examples of the Hermitian method	168
I. Inhomogeneous problem of the second order	168
II. An eigenvalue problem	170
2.7. A Hermitian method for linear boundary-value problems	171
§ 3. Some theoretical aspects of the finite-difference methods	173
3.1. Solubility of the finite-difference equations and convergence of iterative solutions	173
3.2. A general principle for error estimation with the finite-difference methods in the case of linear boundary-value problems	176
3.3. An error estimate for a class of linear boundary-value problems of the second order	177
3.4. An error estimate for a non-linear boundary-value problem	179
§ 4. Some general methods	181
4.1. Examples of collocation	181
4.2. An example of the least-squares method	184
4.3. Reduction to initial-value problems	184
4.4. Perturbation methods	186
4.5. The iteration method, or the method of successive approximations .	189
4.6. Error estimation by means of the general iteration theorem	191
4.7. Special case of a non-linear differential equation of the second order	192
4.8. Examples of the iteration method with error estimates	195
I. A linear problem	195
II. Non-linear oscillations	198
4.9. Monotonic boundary-value problems for second-order differential equations	200

	page
§ 5. Ritz's method for second-order boundary-value problems	202
5.1. EULER's differential equation in the calculus of variations	202
5.2. Derivation of EULER's conditions	203
5.3. The Ritz approximation	207
5.4. Examples of the application of RITZ's method to boundary-value problems of the second order	209
I. A linear inhomogeneous boundary-value problem	209
II. An eigenvalue problem	210
III. A non-linear boundary-value problem	212
§ 6. Ritz's method for boundary-value problems of higher order	213
6.1. Derivation of higher order Euler equations	213
6.2. Linear boundary-value problems of the fourth order	216
6.3. Example	219
6.4. Comparison of RITZ's method with the least-squares process	220
§ 7. Series solutions	222
7.1. Series solutions in general	222
7.2. Power series solutions	223
7.3. Examples	224
§ 8. Some special methods for eigenvalue problems	227
8.1. Some concepts and results from the theory of eigenvalue problems .	228
8.2. The iteration method in the general case	230
8.3. The iteration method for a restricted class of problems	231
8.4. Practical application of the method	233
8.5. An example treated by the iteration method	235
8.6. The enclosure theorem	236
8.7. Three minimum principles	239
8.8. Application of RITZ's method	241
8.9. TEMPLE's quotient	245
8.10. Some modifications to the iteration method	249
8.11. Miscellaneous exercises on Chapter III	251
8.12. Solutions	253

Chapter IV

Initial- and initial-/boundary-value problems
in partial differential equations

The need for a sound theoretical foundation	260
§ 1. The ordinary finite-difference method	262
1.1. Replacement of derivatives by difference quotients	263
1.2. An example of a parabolic differential equation with given boundary values	265
1.3. Error propagation	268
1.4. Error propagation and the treatment of boundary conditions . .	272
1.5. Hyperbolic differential equations	277
1.6. A numerical example	278
1.7. Graphical treatment of parabolic differential equations by the finite-difference method	280
1.8. The two-dimensional heat equation	284
1.9. An indication of further problems	286
§ 2. Refinements of the finite-difference method	286
2.1. The derivation of finite equations	287
2.2. Application to the heat equation	288

	page
2.3. The "Hermitian" methods	291
2.4. An example	293
§ 3. Some theoretical aspects of the finite-difference methods	295
3.1. Choice of mesh widths	295
3.2. An error estimate for the inhomogeneous wave equation	296
3.3. The principle of the error estimate for more general problems with linear differential equations	299
3.4. A more general investigation of error propagation and "stability" .	301
3.5. An example: The equation for the vibrations of a beam	303
§ 4. Partial differential equations of the first order in one dependent variable	305
4.1. Results of the theory in the general case	306
4.2. An example from the theory of glacier motion	308
4.3. Power series expansions	310
4.4. Application of the finite-difference method	311
4.5. Iterative methods	315
4.6. Application of HERMITE's formula	318
§ 5. The method of characteristics for systems of two differential equations of the first order	318
5.1. The characteristics	319
5.2. Consistency conditions	321
5.3. The method of characteristics	323
5.4. Example	324
§ 6. Supplements	329
6.1. Monotonic character of a wide class of initial-/boundary-value problems in non-linear parabolic differential equations	329
6.2. Estimation theorems for the solutions	331
6.3. Reduction to boundary-value problems	333
6.4. Miscellaneous exercises on Chapter IV	334
6.5. Solutions	336

Chapter V

Boundary-value problems in partial differential equations

§ 1. The ordinary finite-difference method	342
1.1. Description of the method	343
1.2. Linear elliptic differential equations of the second order	344
1.3. Principle of an error estimate for the finite-difference method .	348
1.4. An error estimate for the iterative solution of the difference equations	353
1.5. Examples of the application of the ordinary finite-difference method	355
I. A problem in plane potential flow	355
II. An equation of more general type	357
III. A differential equation of the fourth order	359
1.6. Relaxation with error estimation	360
1.7. Three independent variables (spatial problems)	368
1.8. Arbitrary mesh systems	370
1.9. Solution of the difference equations by finite sums	371
1.10. Simplification of the calculation by decomposition of the finite-difference equations	372
§ 2. Refinements of the finite-difference method	375
2.1. The finite-difference method to a higher approximation in the general case	375

	page
2.2. A general principle for error estimation	377
2.3. Derivation of finite expressions	378
2.4. Utilization of function values at exterior mesh points	380
2.5. Hermitian finite-difference methods (<i>Mehrstellenverfahren</i>)	384
2.6. Examples of the use of Hermitian formulae	387
2.7. Triangular and hexagonal mesh systems	389
2.8. Applications to membrane and plate problems	391
§ 3. The boundary-maximum theorem and the bracketing of solutions	396
3.1. The general boundary-maximum theorem	396
3.2. General error estimation for the first boundary-value problem	399
3.3. Error estimation for the third boundary-value problem	401
3.4. Examples	403
3.5. Upper and lower bounds for solutions of the biharmonic equation	406
§ 4. Some general methods	407
4.1. Boundary-value problems of monotonic type for partial differential equations of the second and fourth orders	408
4.2. Error distribution principles. Boundary and interior collocation	409
4.3. The least-squares method as an interior and a boundary method	414
I. Interior method	414
II. Boundary method	417
4.4. Series solutions	418
4.5. Examples of the use of power series and related series	419
4.6. Eigenfunction expansions	422
§ 5. The Ritz method	425
5.1. The Ritz method for linear boundary-value problems of the second order	425
5.2. Discussion of various boundary conditions	427
5.3. A special class of boundary-value problems	429
5.4. Example	429
5.5. A differential equation of the fourth order	431
5.6. Direct proof of two minimum principles for a biharmonic boundary-value problem	432
5.7. More than two independent variables	435
5.8. Special cases	437
5.9. The mixed Ritz expression	438
§ 6. The Trefftz method	441
6.1. Derivation of the Trefftz equations	441
6.2. A maximum property	443
6.3. Special case of the potential equation	445
6.4. More than two independent variables	446
6.5. Example	447
6.6. Generalization to the second and third boundary-value problems	449
6.7. Miscellaneous exercises on Chapter V	451
6.8. Solutions	454

Chapter VI

Integral and functional equations

§ 1. General methods for integral equations	467
1.1. Definitions	467
1.2. Replacement of the integrals by finite sums	469

	page
1.3. Examples	470
I. Inhomogeneous linear integral equation of the second kind	470
II. An eigenvalue problem	472
III. An eigenvalue problem for a function of two independent variables	474
IV. A non-linear integral equation	477
1.4. The iteration method	478
1.5. Examples of the iteration method	479
I. An eigenvalue problem	479
II. A non-linear integral equation	480
III. An error estimate for a non-linear equation	481
1.6. Error distribution principles	482
1.7. Connection with variational problems	484
1.8. Integro-differential equations and variational problems	489
1.9. Series solutions	492
1.10. Examples	496
I. An inhomogeneous integro-differential equation	496
II. A non-linear integral equation	497
§ 2. Some special methods for linear integral equations	498
2.1. Approximation of kernels by degenerate kernels	498
2.2. Example	500
2.3. The iteration method for eigenvalue problems	502
§ 3. Singular integral equations	504
3.1. Smoothing of the kernel	504
3.2. Singular equations with Cauchy-type integrals	505
3.3. Closed-form solutions	510
3.4. Approximation of the kernel by degenerate kernels	511
§ 4. Volterra integral equations	512
4.1. Preliminary remarks	512
4.2. Step-by-step numerical solution	513
4.3. Method of successive approximations (iteration method)	516
4.4. Power series solutions	518
§ 5. Functional equations	519
5.1. Examples of functional equations	519
5.2. Examples of analytic, continuous and discontinuous solutions of functional equations	522
5.3. Example of a functional-differential equation from mechanics	525
5.4. Miscellaneous exercises on Chapter VI	527
5.5. Solutions	528

Appendix

Table I. Approximate methods for ordinary differential equations of the first order: $y' = f(x, y)$	536
Table II. Approximate methods for ordinary differential equations of the second order: $y'' = f(x, y, y')$	537
Table III. Finite-difference expressions for ordinary differential equations .	538
Table IV. Euler expressions for functions of one independent variable . .	540
Table V. Euler expressions for functions of two independent variables .	541
Table VI. Stencils for the differential operators ∇^2 and ∇^4	542

Table VII.	Catalogue of examples treated	547
Table VIII.	Taylor expansion of a general finite expression involving the differential operators ∇^2 and ∇^4 for a square mesh	552
Table IX.	Taylor expansion of a general finite expression involving the differential operators ∇^2 and ∇^4 for a triangular mesh	553
Table X.	Taylor expansion of a general finite expression involving the differential operators ∇^2 and ∇^4 for a cubical mesh.	554
Author index		555
Subject index.		559